Journal of Mathematics and Statistics 3 (3) : 83-87, 2007
ISSN 1549-3644

Dynamics Modelling of a Plate Vibrating in a Perfect Fluid

[1]Bouarroudj Nadra and [2]Belaib Lekhmissi
[1]Department of Mathematics and Informatics E.N.S.E.T Oran Algeria
[2]Department of Mathematics Faculty of Sciences University d'Oran Es-Sénia Oran Algeria

Abstract: We deal with the interaction problem of a plate vibrating within a perfect fluid. We establish the equations describing the dynamics behaviour of the plate using the general equations of the elasto-dynamic. The fluid flow described by the equation of Euler's low amplitude. We presented results of the existence, the unicity and the regularity of the problem verified by the plate as well as by the fluid We used the integral equations for the numerical resolution of the problem that allowed us to determine the coupling term between the fluid and the plate. The numerical results were obtained using finite element method coupled with an implicit diagram in time .

Key words: Interaction, regularity, differential equations, dynamics, finite element method, implicit diagram, integral equation, perfect fluid .

INTRODUCTION

We study the interaction problem of a plate vibrating within a perfect fluid. The present work is inspired from the method introduced by P.G Ciarlet and P.H Destuyder while using a simplified version in order to get the equations describing the dynamic behaviour of the plate from the general equations of the elasto - dynamic.

We consider a plate taking up a domain G from a refined space R^3, we denote by h the plate thickness and by θ its average surface such that $G = \theta x \left]\frac{-h}{2}, \frac{h}{2}\right[$ and $\Gamma^+ = \theta x \left\{\frac{h}{2}\right\}$ the upper face, $\Gamma^- = \theta x \left\{\frac{-h}{2}\right\}$ the lower face of the plate and Γ_0 the lateral side of the plate. We suppose that the plate is embedded along Γ_0 and vibrate under the action of volumic forces of density f and surface forces of density g^+ on Γ^+ and g^- on Γ^- .

We suppose that the plate is constituted of an elastic, homogeneous Isotrope material of density ρ, a Young module E and a Poisson coefficient γ, without hypotheses on the relative measurements of θ and h , the couple (σ, u) of the constraints and the displacements of components σ_{ij} and U_i on $Gx\Re^+$ $i, j = 1,2,3$ is determined in the frame of the linear elasticity by the system to the partial derivatives equations :

Conservation of the movement quantity
$\rho\partial_i U_i - \partial_j\sigma_{ij} = f_i$
Law of Hook

$$\frac{(1+\gamma)}{E}\sigma_{ij} - \frac{\gamma}{E}\sigma_{pp}\delta_{ij} = \varepsilon_{ij}(U)$$
$$\varepsilon_{ij}(U) = \frac{(\partial_j U_i + \partial_i U_i)}{2} \qquad (1.1)$$

Conservation of the kinetic moment
$\sigma_{ij} = \sigma_{ji}$ in $Gx\Re^+$
Embedding condition
$U_i = 0$ over $\Gamma_0 x \Re^+$
Condition of free boundary
$\sigma_{ij}.\eta_j / \Gamma^+ = g^+$
η : does normal to ∂G oriented toward the outside

With the Latin index varying from 1 to 3 and Greek of 1 to 2 . the convention of the summation on repeated indexes is adopted .

If the thickness of the plate is negligible compared to the other dimensions we can construct a bi-dimensional problem where the parameters are defined on the average surface and that solved, allows to calculate an approximation of the three-dimensional parameters, for it, we introduce:

Corresponding Author: Bouarroudj Nadra, Department of Mathematics and Informatics E.N.S.E.T Oran Algeria

$\sum = \{ \text{r tensor defined on G}, \ \tau_{ij} \in L^2(G), \tau_{ij} = \tau_{ji} \}$

$V = \{ \text{U vector defined on G}, U = 0 \ \text{on} \ \Gamma_0 \ \text{and} \ V_i \in H^1(G) \}$

The system(1.1) becomes: To find $(\sigma, U) \in \Sigma x V$ as

$$\begin{cases} \int_G \left(\frac{(1+\gamma)}{E} \sigma_{ij}\tau_{ij} - \frac{\gamma}{E}\sigma_{pp}\tau_{pp} \right) \partial G = \int_G \varepsilon_{ij}(U)\tau_{ij}\partial G \\ \int_G \left(\rho\partial_i^2 U_i V_j + \sigma_{ij}\varepsilon_{ij}(U) \right) \partial G = \int_G f_i V_i \partial G + \int_{\Gamma^{\pm}} g_i^{+} V_i \partial\Gamma^{\pm} \end{cases} \quad (1.2)$$

We are going to make the usual hypothesis in the theory of the plates that allow to reach the model of Kirchoff-Love and to arrive at the bi-dimensional system. There are two types of hypothesis:

Mechanical hypothesis: We suppose that the components of the constraints of shearing and pinching are negligible compared to the tangential constraints that is to say:

$$\sigma_{\alpha 3} = \sigma_{3\alpha} = \sigma_{33} \ , \ \alpha = 1,2 \quad (1.3)$$

Geometric hypothesis: We suppose that the displacements are of Kirchoff-Love, namely that the associated deformations verify:

$$\begin{cases} \varepsilon_{\alpha 3}(V) = 0 \ , \ \alpha = 1,2 \\ \varepsilon_{33}(V) = \partial_3 V_3 = 0 \end{cases} \quad (1.4)$$

While integrating (1.4) we can explicit the form of a displacement of Kirchoff-Love given by:

$$\begin{cases} V_3(x_1, x_2, x_3) = v_3(x_1, x_2) \\ V_\alpha(x_1, x_2, x_3) = v_\alpha(x_1, x_2) - x_3\partial_\alpha v_3(x_1, x_2) \ , \ \alpha = 1,2 \end{cases} \quad (1.5)$$

The conditions of embedding show that:

$$\begin{cases} v_3 = \partial_\eta v_3 = 0 \ \text{sur} \, \partial\theta \\ v_\alpha = \partial_\alpha v_3 = 0 \ \text{sur} \, \partial\theta \end{cases} \quad (1.6)$$

We define:

$\sum' = \{ \tau = (\tau_{\alpha\beta}) \ \text{tensor defined on G}, \ \tau_{\alpha\beta} \in L^2(G) \}$

$V = \{ \text{V vector defined on G}, \ V_\alpha \in H^1(G), \text{verifying} \, (1.1) \}$

We obtain the system: To find $U(.,t) \in V$ for all t □0 and $\sigma(., t) \in \Sigma'$ for all t □0 as:

$$\begin{cases} \int_G \left(\frac{(1+\gamma)}{E} \sigma_{\alpha\beta}\tau_{\alpha\beta} - \frac{\gamma}{E}\sigma_{\alpha\alpha}\tau_{\beta\beta} \right) \partial G = \int_G \varepsilon_{\alpha\beta}(V)\tau_{\alpha\beta}\partial G \\ \int_G \left(\rho\partial_i^2 U_i V_j + \sigma_{ij}\varepsilon_{ij}(U) \right) \partial G = \int_G f_i V_i \partial G + \int_{\Gamma^{\pm}} g_i^{+} V_i \partial\Gamma^{\pm} \end{cases} \quad (1.7)$$

By using (1.5),(1.6) and integrating over x_3 and dividing by h the relation (1.7) becomes (1,8)

$$\begin{cases} \int_G m_{\alpha\beta}\varepsilon_{\alpha\beta}\partial\theta + \frac{h^2}{12}\int_\theta n_{\alpha\beta}\partial_{\alpha\beta}v_3\partial\theta + \int_\theta \rho\partial_i^2 u_\alpha v_\alpha \partial\theta + \ldots \\ \qquad \frac{h^2}{12}\int_\theta \rho\partial_i^2\partial_\alpha u_3\partial_\alpha v_3\partial\theta + \int_\theta \rho\partial_i^2 u_3 v_3\partial\theta = \\ \int_\theta f_\alpha v_\alpha\partial\theta + \frac{1}{h}\int_\theta (g_\alpha^+ + g_\alpha^-) v_\alpha\partial\theta + \int_\theta f_3 v_3\partial\theta + \frac{1}{h}\int_\theta (g_3^+ + g_3^-) v_3\partial\theta + \\ \int_\theta f_\alpha\partial_\alpha v_3\partial\theta - \frac{1}{12}\int_\theta (g_\alpha^+ + g_\alpha^-)\partial_\alpha v_3\partial\theta \end{cases}$$

$$m_{\alpha\beta} = \frac{E}{1-\gamma^2}\left((1-\gamma)\varepsilon_{\alpha\beta}(u) + \gamma\varepsilon_{\mu\mu}(u)\delta_{\alpha\beta} \right)$$

are membrane efforts

$$n_{\alpha\beta} = \frac{E}{1-\gamma^2}\left((1-\gamma)\partial_{\alpha\beta}u_3 + \gamma\Delta u_3\delta_{\alpha\beta} \right)$$

are bending efforts

By taking successively $v_3 = 0$ then v=0, the problem (1.8) gets uncouples into an elastic plane problem and a bending problem of the 4th order.

Elastic plane problem

$$\int_\theta \frac{E}{1-\gamma^2}\left((1-\gamma)\varepsilon_{\alpha\beta}(u)\varepsilon_{\alpha\beta}(v) + \gamma\varepsilon_{\mu\mu}(u)\varepsilon_{\beta\beta}(v)\partial\theta + \int_\theta \rho\partial_i^2 u_3 v_3\partial\theta \right. =$$

$$= \int_\theta f_\alpha v_\alpha\partial\theta + \frac{1}{h}\int_\theta \left(g_\alpha^+ + g_\alpha^- \right) v_\alpha\partial\theta$$

Bending problem

$$D\int_\theta \Delta u_3\Delta v_3\partial\theta + \frac{h^3}{12}\int_\theta \rho\partial_i^2\partial_\alpha u_3.\partial_\alpha v_3\partial\theta + h\int_\theta \rho\partial_i^2 u_3 v_3\partial\theta =$$

$$h\int_\theta f_\alpha\partial_\alpha v_\alpha\partial\theta - \frac{h}{12}\int_\theta \left(g_\alpha^+ - g_\alpha^- \right)\partial_\alpha v_3\partial\theta + h\int_\theta f_3 v_3\partial\theta + \int_\theta \left(g_3^+ + g_3^- \right) v_3\partial\theta$$

Plate fluid interaction: We immerse our plate in a fluid and we study the interaction of the plate-fluid system We suppose that the plate is without thickness with respect to the fluid that occupies the fissured domain $\Re^3 - \theta$. The movement of a fluid is one of flow, it is known in every x point of Ω and at all times t, through the determination of the following quantities, The density ρ of the fluid.

* The $V(x,t)$ speed of the fluid particle
* The pressure $p(x,t)$

We suppose that the fluid is perfect, incompressible of density ρ that equals a constant ρ_0 at x and t, and that is the density of the fluid at stability. The movement of the fluid is determined by the equations of the movements of low amplitude of Euler:

* Equation of mass conservation: (condition of incompressibility)
  $\mathrm{div}(v) = 0$ in $\forall t \succ 0$
* Conservation equation of the quantity of movement
  $\partial_i(\rho v) + \nabla p = 0$ in Ω , $\forall t \succ 0$
* Condition at infinity
  $\lim v(x,t) = 0 \ \text{for} \ |x| \to \infty \ \forall t \succ 0$
* Condition of contact
  $v.e|^{+} = v.e|^{-} = \partial_i u_3 \quad \forall t \succ 0$

The movement of the fluid is irrotational, the speed V thus derives from a potential q

$$v = \nabla q \ \ in \ \ \Omega \quad \forall t \succ 0$$

By reporting this relation in the previous equations we get:

$\Delta q = 0$

$\nabla q.e\big|^{+} = \nabla q.e\big|^{-} = \partial_i u$ on $\Omega x \,]0\,,\,\infty[$

$\lim \nabla q(x,t) = \lim \partial_i \nabla q(x,t) = 0$ for $|x| \to \infty$

$\rho_0 \partial_i q + p = p_0$

p_0 = Pressure in the fluid at rest

The forces exerted by the fluid on the plate are going to be solely surfacique and are given by the densities:

$g^{+} = -p^{+}.e$

$g^{-} = p^{-}.e$

Which translate the equality of the normal constraints to the fluid interface plates, they are normal to the surface of the plate. The density of forces exerting on the plate is:

$g^{+} + g^{-} = [p]e = -\rho_0 \partial_t [q]$

Equations describing the vibration of a plate in a perfect fluid: Due to the previous relations the main parameters are going to be the potential q (., t) defined for all t on Ω ,and the displacement normal u (., t) belonging for all t in the space:

$V = \{ v \in H^2(\theta) \text{ defined on } \overline{\theta}, v = \partial_\alpha v = 0 \text{ on } \partial\theta \}$

(We refer to [4] for the Hilbertian properties for it)

These parameters are solutions of the system

To find u (., t)$\in$ V for all t$\succ 0$

(p1)

$$\left\{\begin{array}{lll} a(u(t),v) + \partial_t^2 b(u(t),v) = -\rho_0 \int_\theta \partial_t [q].v\partial\theta & , & \forall v \in V, \forall t \succ 0 \\ \Delta q = 0 & ,\text{in } \Omega = R^3 - \overline{\theta} & \\ \nabla q.e\big|^{+} = \nabla q.e\big|^{-} = \partial_t u & ,\text{ at } \partial\Omega = \theta & \end{array}\right\}$$

a(u(t),v) describes the rigidity of the plate at the given bending by:

$$a(u(t),v) = D\int_\theta ((1-\gamma)\partial_{\alpha\beta} u(t)\partial_{\alpha\beta} v + \gamma\Delta u(t).\Delta v)\partial\theta$$

b(u(t),v) describes the inertia forces of the plate given by:

$$b(u(t),v) = \frac{\rho_0 h^3}{12}\int_\theta \partial_\alpha u(t)\partial_\beta v\,\partial\theta + \rho_0 h\int_\theta u(t).v\,\partial\theta$$

$$D = \frac{\rho h^3}{12(1-\gamma^2)}$$

the term $\rho_0 \int_\theta \partial_t [q]\partial\theta$ describes the coupling between the fluid and the plate

Since the potential q that verifies a problem at the exterior limits ,it is not easy to solve the system (p1) within the conventional methods, for this we use integral equations to bring back the problem on the average surface of the plate and that allows to calculate the potential jump.

Position of the problem: The potential is the solution of the following equations:

$$(p2)\ \left\{\begin{array}{l} \Delta q = 0 \text{ on } \Omega \\ \nabla q.e\big|^{+} = \nabla q.e\big|^{-} = w \text{ in } \partial\Omega = \theta \end{array}\right.$$

This problem is of Neumann type and the existence requires a condition of orthogonality on data and this condition is verified.

Theorem: Given w in $(H_{00}^{1/2}(\theta))'$ the problem (p2) has a unique solution in $W_0^1(\Omega)$

Demonstration: The problem (p2) is equivalent to the variational problem

To find q defined in Ω as

$$(p3)\ \int_\theta \nabla q \nabla \psi\,\partial\Omega = \int_\theta w[\psi]\partial\Omega \quad ,\forall \psi \in W_0^1(\Omega)$$

$W_0^1(\Omega)$ is an Hilbert's space

The bilinear form at left is continuous, coercive on $W_0^1(\Omega)$. The linear form at right is continuous on $W_0^1(\Omega)$ according to the theorem of Lax-Milgram the problem (p3) admits an unique solution in $W_0^1(\Omega)$ where the existence and the unicity of the solution of the potential problem (p2)

We introduce a linear operator A independent of the time defined on $(H_{00}^{1/2}(\theta))'$ to values in $H_{00}^{1/2}(\theta)$ by Aw=[q] where q is solution of the problem (p2).

Properties of the operator A

Theorem: The operator A is an isomorphism of $(H_{00}^{1/2}(\theta))'$ in $H_{00}^{1/2}(\theta)$

Theorem: The operator A allows to define a bilinear form $C(w,v)$ defined by

$$C(w,v) = \int_\theta Aw.v\partial\theta$$

symmetrical definite positive on $(H_{00}^{1/2}(\theta))' x H_{00}^{1/2}(\theta)$

We call $c(.)$ the added mass form or the form of coupling fluid plate

Analysis of the verified problem by the bending u of the plate: Now ,We consider the problem of bending of the thin plate. In the absence of forces, the plate is represented by θ opened from $\Re^2$, We are interested in the calculation of the displacement u according to the perpendicular to θ Since A is independent of the time one has: $\partial_t [q] = \partial_t Aw = \partial_t A \partial_t u = \partial_t^2 Au$

The problem of bending takes the following form:

Find u in $\theta x\,]0\,,T\,[$ with value in $\Re$ solution of the equations:

$$(p4)\begin{cases} \partial_t^2(-\rho\frac{h^2}{12}\Delta u + phu + \rho_0 Au) + D\Delta^2 u = 0 \text{ in } \theta x\,]0\,,\,T[\\ \text{with the conditions to the limits} \\ u = \nabla u.e = 0 \text{ in } \partial\theta x\,]0\,,T[\\ \text{and the initial conditions} \\ u(0,x) = u_0(x)\,,\ \partial_t u(0,x) = u_1(x) \text{ in } \theta \end{cases}$$

Theorem: Given u_0 in $H_0^2(\theta)$ and u_1 in $H_0^1(\theta)$ the problem of bending admits an unique solution in

$$C^0(0,T;H_0^2(\theta)) \cap C^1(0,T;H_0^1(\theta))$$

Demonstration: The problem of bending is equivalent to the following variational problem:
Find $u:[0,T] \to H_0^2(\theta)$ verifying

$$a(u(t),v) + \partial_t^2 b(u(t),v) = 0 \quad \forall v \in H^2(\theta)\,,\ \forall t \succ 0$$
$$u(0\,,\,x) = u_0(x)$$
$$\partial_t u(0\,,\,x) = u_1(x)$$

where

$$b(u(t),v) = \rho_0 \frac{h^3}{12}\int_\theta \Delta u(t)\Delta v \partial\theta + \rho_0 h \int_\theta u(t).v\partial\theta + $$
$$+ \rho_0 \int_\theta Au(t).v\partial\theta$$

The operator A being symmetric, continuous and definite positive .The expression $b(u(t),v)$ is a scalar product on $H_0^1(\theta)$ equivalent to the usual scalar product. The bilinear form $a(u(t),v)$ is symmetric, continuous and coercive on $H_0^2(\theta)xH_0^2(\theta)$. therefore under the hypothesis u_0 in $H_0^2(\theta)$ and u_1 in $H_0^1(\theta)$ according to the theorem of Lax-Milgram , the problem variational (p5) admits an unique solution $in\ H_0^2(\theta)$ where the existence and the uniqueness of the solution of the problem (p4)

Integral representation of the potential problem

Expression of the added mass: One of the difficulties of the bending problem is the absence of an explicit expression of the bilinear form c(.,.) We use the integral equations [3] to bring the outside problem (p2) to a a problem on θ ,and that allows us not to have an expression of the form c (.,.) but to be able to calculate its value We consider $V = \nabla q$ and calcul $\Delta V = \nabla\Delta q$ in the sens of the distribution on $\Re^3$ and using the elementary solution of the equation of Laplace in $\Re^3$ we obtain (with $V = \nabla q$)

$$V(x) = -\text{rot}\int_\theta \frac{1}{4\pi|x-y|}[ex V]\partial\theta = -\text{rot}\int_\theta \frac{1}{4\pi|x-y|}[ex\nabla q]\partial\theta$$
$$= -\text{rot}\int_\theta \frac{1}{4\pi|x-y|} ex[\nabla q]\partial\theta$$

Let ψ be regular on the boundary of θ ,we have

$$\int_\theta V.e\psi\partial\theta = \int_\theta -\text{rot}V.e\psi\partial\theta = \int_\theta e\nabla\psi.V\partial\theta = \int_\theta \frac{1}{4\pi|x-y|}\nabla\varphi\nabla\psi\partial\theta = \int_\theta w\psi$$

with $\varphi = Aw$

To get $\varphi = Aw$ it is sufficient to solve the integral equation:

For a given w in $(H_{00}^{1/2}(\theta))'$ find φ in $H_{00}^{1/2}(\theta)$ satisfying:

$$\iint_\theta \frac{1}{4\pi|x-y|}\nabla\varphi\nabla\psi\partial\theta = \prec w\,,\,\psi \succ$$

Where $\prec\,,\,\succ$ designates the duality between $(H_{00}^{1/2}(\theta))'$ and $H_{00}^{1/2}(\theta)$

Once φ obtained, we calculate $c(w,v) = \int_\theta \varphi v \partial\theta$

Theorem: For $w \in (H_{00}^{1/2}(\theta))'$ the problem Find $\varphi \in H_{00}^{1/2}(\theta)$

$$\iint_\theta \frac{1}{4\pi|x-y|}\nabla\varphi\nabla\psi\partial\theta = \prec w\,,\,\psi \succ$$

admits an unique solution

Demonstration: By using the theorem of Lax-Milgram the bilinear form is continuous coercive on $H_{00}^{1/2}(\theta)$ and that the linear form at right and linear continuous where the existence and the unicity of the problem of the integral equation.

Approximation

Problem of the integral equation: To approach the problem of the integral equation we are brought to use finite elements [2] of class C^0 , we write the discreet problem that amounts to a matrix linear system

[K] X = F

[K] global rigidity matrix

$$K(i,j) = \iint_\theta \frac{1}{4\pi|x-y|}\nabla\lambda_i\nabla\lambda_j\partial\theta\partial\theta$$

$$F(j) = \int_\theta w\lambda_j\partial\theta$$

$$X(i) = \varphi_h(m_i)$$

i/T = number of the i summit of the T triangle

$\lambda_{iLT} \in P_1$

To calculate the elementary matrix [ke] we need to calculate the elementary integrals

$$\iint_{\theta} \frac{1}{4\pi|x-y|} \nabla\lambda_i \nabla\lambda_j \partial\theta\,\partial\theta$$

Several methods have been tested for the calculation of these integrals it was evident from this analysis that one of the methods consists in doing the first integration analytically by using the method of de Hoop and numerically the second by using a numerical formula of integration if the two triangles have an non-empty intersection and to approach the two integrals if the two integrals are disconnected

We write therefore

$$ZA = \frac{1}{4\pi}\int_K \mathrm{Funct}(x)\partial K, \quad \mathrm{Funct}(x) = \int_L \frac{1}{|x-y|}\partial L$$

Problem of bending: For the approximation of the bending problem (pf) we are brought to use elements finite[2] of class C^1 for example the finite element of Argyris coupled with an implicit diagram in time .

CONCLUSION

Our work is a numerical attempt at resolution of the equation of coupling fluid-plate which occupies an important place in the practical domain.

We elaborated a program in language Fortran allowing to calculate the solutions approached the problem posed .A this effect; we used the method of Cholesky for the variable of the space and the method of Newmark for the variable of time

We made a first digital test for the problem of the integral equation for various choices of w:

$$w=1,\ w=x,\ \sin(x)e^y,\quad w=\delta(1/2, 1/2),$$

$$w = \begin{cases} 1 & \text{if } x \ge 1/2 \\ 0 & \text{if not} \end{cases}$$

We obtain a convergence according to the meshing.
Curves are close relations for the regular functions .
On get back a good estimate of the solution .

$$\text{for}\quad w=\delta(1/2, 1/2),\ w = \begin{cases} 1 & \text{if } x \ge 1/2 \\ 0 & \text{if not} \end{cases}$$

one notice that the values in the point $(1/2, 1/2)$ and are a little taken away because of the singular point who corresponds in the Second member w.

To improve the convergence it is necessary to refine the meshing at the level of the peculiarity for our program we used a uniform meshing .let us hope to generalize it afterward and to take a not uniform meshing .

The second test is for the problem of flexion. we tested the problem corresponding successively to three deflections:

$$u(x,y,t) = x^2(x-1)^2 y^2(y-1)^2(1+t^2)$$

$$u(x,y,t) = y^2(y-1)^2 \sin^2(\pi x)(1+t^2)$$

$$u(x,y,t) = y^2(y-1)^2 \sin^2(2\pi x)(1+t^2)$$

We notice that we have the convergence of the solution calculated towards
the exact solution, curves are close relations, we get back a good estimate of the solution

REFERENCES

1. Adams Robert , 2003 :Sobolev spaces (Pure & applied mathematics series) Hardback
2. Ciarlet P.G ,1979 : The finite element method for elliptic problems, North Holland
3. Nedelec J.C ,1982 : Integral equation and operator theory . Vol.5
4. Peter Fillmore, Nigel Higson ,1984 : A Hilbert Space Problem . Vol 91, n°9