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On the Transformation T_{+m} Due Gray and Clark

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Abstract: We determine the values of the integer m for which the parametric transformation T_{+m} due Gray and Clark is well conditionned. This process of acceleration being quasilinear transformation, we use an adequate definition of the condition numbers that apply to the real sequences of linear convergence. The results obtained for this set are enough meaning.

Key words: Conditions numbers, linear convergence, logarithmic convergence, punctual conditioning, asymptotic conditioning.

INTRODUCTION

Nonlinear sequences transformations are generally used for solve the extrapolation of the limit or to accelerate slowly convergent sequences. One can also use them to sum the divergent series[1, 11]. But most of the time, the implementation of these transformations provides us with sequences whose terms are sullied with errors due to the limits imposed by the capacities of the computer on the one hand and by the truncation errors on the other hand.

To these errors, we can add the conditioning of the transformation which is used. It is thus significant to make a study of conditioning. We are interested in this work in the conditioning of the transformation T_{+m} of Gray and Clark, which generalizes the famous Δ^2-Aitken's process.

For that we use the definition[4] of the number of conditioning when the transformation considered is quasi-linear. Thereafter, we will discuss the values of the integer m for which we obtain the best conditioned transformation.

The first generalization of Aitken's Δ^2 process is T_{+m} transformation due to Gray[8] and Clark]. Let m be a strictly positive integer and consider the sequence

$$T_{+m}^{(n)} = T_{+m}(S_n) = S_n - \frac{\Delta S_n}{\Delta S_{n+m} - \Delta S_n} \cdot (S_{n+m} - S_n)$$

where $(S_n)_{n\geq 0}$ is a sequence or an infinite series.

For $m = 1$, Aitken's Δ^2 process is recovered. Acceleration results were given by the previous authors and by Streit[14]. Let $x = (x_k)_k$ be a convergent sequence of a limit x^* and let Φ be a mapping defined on $\mathbb{R}^{p+1}$, where $p \geq 2$. In the paper, we shall consider Φ as a function of $(p+1)$-variables $u_1, u_2, ..., u_p$.

The sequence transformation $\Phi : x \to y$ is defined by

$$y_i = \Phi(x_i, x_{i+1}, ..., x_{i+p}),\ i = 0,1,2,...$$

Let us assume that Φ is function of class C^1 on an open subset A of $\mathbb{R}^{p+1}$.

Definition 1: The transformation Φ is quasilinear if the following properties are satisfied

(a) A property of translativity:

$$\forall (u_0, u_1, ..., u_p) \in A \forall b \in \mathbb{R} :$$
$$\Phi(u_0 + b, u_1 + b, ..., u_p + b) = \Phi(u_0, u_1, ..., u_p) + b$$

(b)A property of homogeneity:

$$\forall (u_0, u_1, ..., u_p) \in A \forall a \in \mathbb{R} :$$
$$\Phi(au_0, au_1, ..., au_p) = a\Phi(u_0, u_1, ..., u_p)$$

Most extrapolation processes are quasilinear transformations such as Aitken's Δ^2 process[4], Wynn's epsilon-algorithm[5,12] and more generally the E-algorithm[7] (under certain assumption), the Θ-algorithm of Brezinski[6] and many other processes.

Now, we consider the mapping Φ defined on $\mathbb{R}^{m+2}$ by

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$$\Phi_m(u_0,u_1,\ldots,u_{m+1}) = u_0 - \frac{u_1-u_0}{u_{m+1}-u_m-u_1+u_0}(u_m-u_0)$$

If we apply Φ_{+m} to

$$(S_n, S_{n=1},\ldots,S_{n+m},S_{n+m+1}),$$

we will obtain

$$\begin{aligned}&\Phi_m(S_n,S_{n+1},\ldots,S_{m+m},S_{n+m+1}) = \\ &S_n - \frac{S_{n+1}-S_n}{S_{n+m+1}-S_{n+m}-S_{n+1}+S_n}\cdot(S_{n+m}-S_n) \\ &= S_n - \frac{\Delta S_n}{\Delta S_{n+m}-\Delta S_n}\cdot(S_{n+m}-S_n) = T_{+m}^{(n)}\end{aligned}$$

his map Φ_{+m} is associated with the T_{+m} transformation.

Proposition 1: The transformation T_{+m} of Gray and Clark is a quasilinear transformation.

Proof: For one justification , one can refer to the proof given in[3].
In this work we are interested only in stationary processes defined in the sense of Ortega[9] and Rheinboldt . Let Φ be the transformation defined before by

$$y_i = \Phi(x_i, x_{i+1},\ldots,x_{i+p}),\ i = 0,1,2,\ldots$$

We are concerned with the way by which y_i is changed when we perturb the numbers $x_i, x_{i+1},\ldots,x_{i+p}$. We denote by δx_q the perturbation on the term x_q, $x \in \mathbb{N}$. The resulting perturbation on the term y_i is given by

$$\delta y_i = \delta x_i \frac{\partial\Phi}{\partial u_0}(\xi_i) + \delta x_{i+1}\frac{\partial\Phi}{\partial u_1}(\xi_{i_i}) + \ldots + \delta x_{i+p}\frac{\partial\Phi}{\partial u_p}(\xi_i)$$

where

$$\begin{aligned}&\xi_i = (x_i+\theta_0\delta x_i, x_{i+1}+\theta_1\delta x_{i+1},\ldots,x_{i+p}+\theta_{i+p}\delta x_{i+p}) \\ &0<\theta_j<1,\ j=0,1,\ldots\end{aligned}$$

It follow that

$$|\delta y_i| \le \max(|\delta x_i|,|\delta x_{i+1}|,\ldots,|\delta x_{i+p}|)\cdot\sum_{j=0}^{p}\left|\frac{\partial\Phi}{\partial u_j}(\xi_i)\right|$$

Setting $X_i = (x_i, x_{i+1},\ldots,x_{i+p}) \in \mathbb{R}^{p+1}$, $i = 0,1,2,\ldots$ and evaluating the partial derivatives at the point X_i rather than ξ_i, we can give the following definition.

Definition 2: The punctual condition number i^{th} step of Φ for a sequence x is given by

$$C_i(\Phi,x) = \sum_{j=0}^{p}\left|\frac{\partial\Phi}{\partial u_j}(X_i)\right|, i = 0,1,2\ldots$$

This number is in fact the factor of amplification of the errors made on the term x_i.
In the case where the sequence of the punctual condition numbers converges, then we can define the asymptotic conditioning of the transformation Φ, applied to the sequence x, by

$$C_\infty(\Phi,x) = \lim C_i(\Phi,x)$$

Denoting the transformation defined on $\mathbb{R}^{m+2}$ by Φ_m we have

$$\Phi_m(u_0,u_1,\ldots,u_m,u_{m+1}) = u_0 - \frac{u_1-u_0}{u_{m+1}-u_m-u_1+u_0}\cdot(u_m-u_0)$$

For $m=1$, we find $\Phi_1(u_0,u_1,u_2) = \dfrac{u_0u_2-u_1^2}{u_2-2u_1+u_0}$. This is the transformation that defines the Aitken's Δ^2 process .
For $m=2$, the transformation T_{+2} is

$T_{+2}^{(n)} = \dfrac{S_n\Delta S_{n+2} - S_{n+2}\Delta S_n}{\Delta S_{n+2}-\Delta S_n}$ and the Φ_2 associated transformation is

$$\Phi_2(u_0,u_1,u_2,u_3) = \frac{u_0(u_3-u_2)-u_2(u_1-u_0)}{u_3-u_2+u_1-u_0} = \frac{u_0u_3-u_1u_2}{u_3-u_2+u_1-u_0}$$

Calculus of partial derivatives

$$\frac{\partial\Phi_2}{\partial u_0}(u_0,u_1,u_2,u_3) = \left(\frac{u_3-u_2}{u_1-u_0}\right)\left(\frac{u_3-u_2}{u_1-u_0}+\frac{u_2-u_1}{u_1-u_0}\right):\left(\frac{u_3-u_2}{u_1-u_0}-1\right)^2$$

$$\frac{\partial\Phi_2}{\partial u_1}(u_0,u_1,u_2,u_3) = -\left(\frac{u_3-u_2}{u_1-u_0}\right)\left(\frac{u_2-u_1}{u_1-u_0}+1\right):\left(\frac{u_3-u_2}{u_1-u_0}-1\right)^2$$

$$\frac{\partial\Phi_2}{\partial u_2}(u_0,u_1,u_2,u_3) = -\left(\frac{u_3-u_2}{u_1-u_0}\right)\left(\frac{u_3-u_2}{u_1-u_0}+1\right):\left(\frac{u_3-u_2}{u_1-u_0}-1\right)^2$$

$$\frac{\partial \Phi_2}{\partial u_3}(u_0,u_1,u_2,u_3)=\left(\frac{u_2-u_1}{u_1-u_0}+1\right):\left(\frac{u_3-u_2}{u_1-u_0}-1\right)$$

It is easy to verify that

$$\sum_{i=0}^{3}\frac{\partial \Phi_2}{\partial u_i}(u_0,u_1,u_2,u_3)=1\,.$$

This is a general result. The quasilinear transformations verify this property.

Definition 3: We call LIN the set of linear sequences, that is the set of convergent sequences $x=(x_n)_n$ such that $\exists \rho \neq 0, \rho \in]-1,+1[$ such that

$$\lim_{n\to\infty}\frac{x_{n+1}-x^*}{x_n-x^*}=\rho$$

If $\rho \in]-1,0[$, x is an alternating sequence. We denote this set by LIN^-.

If $x \in]0,1[$, x is monotonic linear sequence and we denote this set by LIN^+.

Proposition 2: If $x=(x_n)_n$ is a sequence of the set LIN, then

$$\lim_{n\to\infty}\frac{x_{n+1}-x^*}{x_n-x^*}=\rho \Rightarrow \lim_{n\to\infty}\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}=\rho$$

Proof

$$=\frac{(x_{n+1}-x^*)}{(x_n-x^*)}\cdot\frac{((x_{n+2}-x^*)/(x_{n+1}-x^*)-1)}{((x_{n+1}-x^*)/(x_n-x^*)-1)}.\lim_n\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}$$
$$=\lim_n\frac{(x_{n+1}-x^*)}{(x_n-x^*)}\cdot\lim_n\frac{((x_{n+2}-x^*)/(x_{n+1}-x^*)-1)}{((x_{n+1}-x^*)/(x_n-x^*)-1)}$$
$$=\rho\cdot\left(\frac{\rho-1}{\rho-1}\right)=\rho$$

Definition 4: Let $x=(x_n)$ be a convergent sequence of a limit x^*. Then a sequence of transformation T that $y_n=T(x_n)$ is said to be regular if the transformed sequence is converging and to the same limit x^*.

Proposition 3: On the set LIN, the quasilinear T_{+m} of Gray and Clark is a regular transformation.

Proof: For one justification , one can refer to the proof given in[10] .

Results in particular cases: For $m=1$, the asymptotic condition numbers

$$C_\infty(\Phi,x)=\left(\frac{1+|\rho|}{1-\rho}\right)^2$$

This last formula was established in[10]]and allows us to verify the work done previously.

We observe that the process of Aitken is well conditioned when applied to elements of LIN^- $(-1<\rho<0)$. However, it is ill conditioned if the asymptotic ratio of the sequence (which we have to accelerate) is close to one.

For m=2, we take the following results established in[3]

First case: x is an alternating sequence $(\rho \in]-1,0[)$ and

$$C_\infty(\Phi_2,x)=\frac{1-\rho^4}{(\rho^2-1)^2}$$

Second case: x is a monotonic sequence $(\rho \in]0,1[)$ and

$$C_\infty(\Phi_2,x)=\frac{1-\rho^4}{(\rho^2-1)^2}$$

Evaluating of the asymptotic condition number in the case where $m\geq 1$. Let Φ_m be the transformation defined on $\mathbb{R}^{m+2}$

$$\Phi_m(u_0,\ldots,u_{m+1})=u_0-\frac{u_1-u_0}{u_{m+1}-u_m-u_1+u_0}(u_m-u_0)$$

Calculus of partial derivatives in general case

$$\frac{\partial \Phi_m}{\partial u_0}(u_0,\ldots,u_{m+1})=\frac{(u_{m+1}-u_m)(u_{m+1}-u_1)}{(u_1-u_0)^2((u_{m+1}-u_m)/(u_1-u_0)-1)^2}$$

$$\frac{\partial \Phi_m}{\partial u_1}(u_0,\ldots,u_{m+1})=-\frac{(u_{m+1}-u_m)(u_m-u_0)}{(u_1-u_0)^2((u_{m+1}-u_m)/(u_1-u_0)-1)^2}$$

$$\frac{\partial \Phi_m}{\partial u_2}(u_0,\ldots,u_{m+1})=0,\ldots,\frac{\partial \Phi_m}{\partial u_{m-1}}(u_0,\ldots,u_{m+1})=0$$

$$\frac{\partial \Phi_m}{\partial u_m}(u_0,\ldots,u_{m+1})=-\frac{(u_{m+1}-u_1)}{(u_1-u_0)^2((u_{m+1}-u_m)/(u_1-u_0)-1)^2}$$

$$\frac{\partial\Phi_m}{\partial u_{m+1}}(u_0,...,u_{m+1}) = -\frac{(u_m - u_0)}{(u_1-u_0)^2((u_{m+1}-u_m)/(u_1-u_0)-1)^2}$$

We apply the transformation Φ_m to terms $x_n, x_{n+1}, ..., x_{n+m}, x_{n+m+1}$ of the sequence $(x_n)_n \in LIN$. The partial derivatives evaluated at the points $x_n, x_{n+1}, ..., x_{n+m}, x_{n+m+1}$ are respectively

$$\frac{\partial\Phi_m}{\partial u_0}(x_n,...,x_{n+m+1}) = \frac{(x_{n+m+1}-x_{n+m})(x_{n+m+1}-x_n)}{(x_{n+1}-x_n)^2((x_{n+m+1}-x_{n+m})/(x_{n+1}-x_n)-1)^2}$$

$$\frac{\partial\Phi_m}{\partial u_1}(x_n,...,x_{n+m+1}) = -\frac{(x_{n+m+1}-x_{n+m})(x_{n+m}-x_n)}{(x_{n+1}-x_n)^2((x_{n+m+1}-x_{n+m})/(x_{n+1}-x_n)-1)^2}$$

$$\frac{\partial\Phi_m}{\partial u_2}(x_n,...,x_{n+m+1}) = 0, ..., \frac{\partial\Phi_m}{\partial u_{m-1}}(x_n,...,x_{n+m+1}) = 0$$

$$\frac{\partial\Phi_m}{\partial u_{m+1}}(x_n,...,x_{n+m+1}) = \frac{(x_{n+m}-x_n)}{(x_{n+1}-x_n)^2((x_{n+m+1}-x_{n+m})/(x_{n+1}-x_n)-1)^2}$$

Using $\rho_n = \frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}$ we can formulate theses partial derivatives as follows

$$\frac{\partial\Phi_m}{\partial u_0}(x_n,...,x_{n+m+1}) = \frac{\left(\prod_{i=1}^{m}\rho_{n+m-i}\cdot\sum_{j=1}^{m}\prod_{i=j}^{m}\rho_{n+m-i}\right)}{\left(\prod_{i=1}^{m}\rho_{n+m-i}-1\right)^2},$$

$$\frac{\partial\Phi_m}{\partial u_1}(x_n,...,x_{n+m+1}) = -\frac{\prod_{i=1}^{m}\rho_{n+m-i}\cdot\left(\sum_{j=2}^{m}\prod_{i=j}^{m}\rho_{n+m-i}+1\right)}{\left(\prod_{i=1}^{m}\rho_{n+m-i}-1\right)^2}$$

$$\frac{\partial\Phi_m}{\partial u_2}(x_n,...,x_{n+m+1}) = 0, \qquad \frac{\partial\Phi_m}{\partial u_{m-1}}(x_n,...,x_{n+m+1}) = 0$$

$$\frac{\partial\Phi_m}{\partial u_m}(x_n,...,x_{n+m+1}) = -\frac{\left(\sum_{j=1}^{m}\prod_{i=j}^{m}\rho_{n+m-i}\right)}{\left(\prod_{i=1}^{m}\rho_{n+m-i}-1\right)^2},$$

$$\frac{\partial\Phi_m}{\partial u_{m+1}}(x_n,...,x_{n+m+1}) = \frac{\left(\sum_{j=2}^{m}\prod_{i=j}^{m}\rho_{n+m-i}\right)}{\left(\prod_{i=1}^{m}\rho_{n+m-i}-1\right)^2}$$

The punctual conditioning of the sequence $x = (x_n)_n$ is

$$C_n(\Phi_m, x) = \sum_{k=0}^{m+1}\left|\frac{\partial\Phi_m}{\partial u_k}(X_n)\right|, \quad n = 0,1,2,...$$
$$X = (x_n, x_{n+1}, ..., x_{n+m+1}) \in \mathbb{R}^{m+2}$$

The asymptotic condition numbers is well defined since the denominator is different from zero. Since $\rho = \lim_{n\to\infty}\rho_n$ it follows that

$$\lim_{n\to\infty}\frac{\partial\Phi_m}{\partial u_0}(x_n,...,x_{n+m+1}) = \frac{\rho^{m+1}(1+\rho+\rho^2+...+\rho^{m-1})}{(\rho^m-1)^2},$$

$$\lim_{n\to\infty}\frac{\partial\Phi_m}{\partial u_1}(x_n,...,x_{n+m+1}) = -\frac{\rho^{m}(1+\rho+\rho^2+...+\rho^{m-1})}{(\rho^m-1)^2}$$

$$\lim_{n\to\infty}\frac{\partial\Phi_m}{\partial u_2}(x_n,...,x_{n+m+1}) = 0, ..., \lim_{n\to\infty}\frac{\partial\Phi_m}{\partial u_{m-1}}(x_n,...,x_{n+m+1}) = 0$$

$$\lim_{n\to\infty}\frac{\partial\Phi_m}{\partial u_m}(x_n,...,x_{n+m+1}) = -\frac{\rho(1+\rho+\rho^2+...+\rho^{m-1})}{(\rho^m-1)^2}$$

$$\lim_{n\to\infty}\frac{\partial\Phi_m}{\partial u_{m+1}}(x_n,...,x_{n+m+1}) = \frac{(1+\rho+\rho^2+...+\rho^{m-1})}{(\rho^m-1)^2}$$

Proposition 4: According to the sign of ρ and the parity of m, the signs of the partial derivatives are given below

m odd; $\rho < 0$ $\quad \frac{\partial\Phi_m}{\partial u_0} > 0 \quad \frac{\partial\Phi_m}{\partial u_1} > 0 \quad \frac{\partial\Phi_m}{\partial u_m} > 0 \quad \frac{\partial\Phi_m}{\partial u_{m+1}} > 0$

$\rho > 0$ $\quad \frac{\partial\Phi_m}{\partial u_0} > 0 \quad \frac{\partial\Phi_m}{\partial u_1} < 0 \quad \frac{\partial\Phi_m}{\partial u_m} < 0 \quad \frac{\partial\Phi_m}{\partial u_{m+1}} > 0$

m even; $\rho < 0$ $\quad \frac{\partial \Phi_m}{\partial u_0} < 0 \quad \frac{\partial \Phi_m}{\partial u_1} < 0 \quad \frac{\partial \Phi_m}{\partial u_m} > 0 \quad \frac{\partial \Phi_m}{\partial u_{m+1}} > 0$

$\rho > 0 \quad \frac{\partial \Phi_m}{\partial u_0} > 0 \quad \frac{\partial \Phi_m}{\partial u_1} < 0 \quad \frac{\partial \Phi_m}{\partial u_m} < 0 \quad \frac{\partial \Phi_m}{\partial u_{m+1}} > 0$

Concluding Results: According to the sign of ρ and the parity of m, we obtain the results:

$\rho < 0$, m odd $C_\infty(\Phi_m, x) = 1$ (well conditioned)

$\rho < 0$, m even $C_\infty(\Phi_m, x) = \frac{(1+\rho^m)}{(1-\rho^m)}$

$\rho > 0$, m odd or even $C_\infty(\Phi_m, x) = \frac{(1+\rho)(1+\rho^m)}{(1-\rho)(1-\rho^m)}$

Numerical applications

In the first table we give the initial sequence's terms computed with Digits:=30, Digits :=9 and Digits :=6.

If we consider the calculus with Digits := 30 as accurate, then that done with Digits := 9 or := 6 will be done with roundoff error which will be a source of a small error. This error will be propagated in the terms of the transformed sequence. The values of transformed sequence are given in the second table. In some cases, we have chosen to study the transformation T_{+m} for the values $m = 1$

Example : Sequences of linear convergence

$$S_n = \sum_{k=0}^{n} (-0.98)^k \ , \ S = \sum_{k=0}^{\infty} (-0.98)^k$$

Initial sequence

	Digits:=30
S	0.505050505050505050505050505050
S_{200}	0.513755650340205623046915475343
S_{202}	0.513410926586733480374257622520
S_{203}	0.496857291945001189233227529931
S_{209}	0.497792609646199382813673677693
S_{210}	0.512163242546724604842599795861

	Digits:=9	Digits:=6
S	0.505050505	0.505050
S_{200}	0.513755650	0.513756
S_{202}	0.513410927	0.513411
S_{203}	0.496857292	0.496857
S_{209}	0.497792610	0.497793
S_{210}	0.512163243	0.512163

Transformed sequence for m :=1

	Digits :=30
S	0.505050505050505050505050505050
W_{200}	0.505050505050505050505050505050
W_{202}	0.505050505050505050505050505051
W_{203}	0.505050505050505050505050505051
W_{209}	0.505050505050505050505050505051
W_{210}	0.505050505050505050505050505050

	Digits:=9	Digits:=6
S	0.505050505	0.505050
W_{200}	0.505050505	0.505050
W_{202}	0.505050505	0.505051
W_{203}	0.505050505	0.505050
W_{209}	0.505050505	0.505051
W_{210}	0.505050506	0.505050

The terms of the sequence $S = (S_n)$ approach to the limit S. Relativement à Digits := 30 , the terms S_n are calculed for Digits :=9, with eight exact digits and with five exact digits for digits:= 6. We remark the roundoff errors generated to ninth digit of initial sequence terms S_n are reproduced also to ninth digit of transformed sequence $W_n = T_{+1}^{(n)}$. This prove the good conditioning of the transformation T_{+1}. This strenghtens the theoritical results that we have obtained $C_\infty(T_{+1}, S) = 1$ with $\rho = (-0.98) < 0$ and $m = 1$ (odd).

CONCLUSION

In this study and within sight of the results obtained, we can say that the transformation of Gray and Clark is well conditioned for m odd on the set of sequences to alternating convergence whereas for the other cases the asymptotic conditioning is expressed according to the integer m. We can plot the graph of the asymptotic conditioning.

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