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One Algebra of New Generalized Functions

Sabra Ramadan
The Kingdom of Saudi Arabia, Ministry of Education, Jazan Teachers College
Department of Mathematics, Jazan-Abu Areesh,P.O. Box 203

Abstract: The space of new Generalized functions $\zeta(\prod(R))$ has been constructed. The operation of associative multiplication Θ has been defined on $\zeta(\prod(R))$.The embedding $J_\pi : \zeta(S(R)) \to \zeta(\prod(R))$ has been constructed.

Key words: Generalized functions, distribution, associative multiplication, semi norms,topology

INTRODUCTION

One of the first problems in distributions theory is how to define the associative multiplication in distribution spaces $S^{/}$ and $D^{/}$. Schwartz(1) demonstrated the impossibility to define Associative Multiplication in such spaces. If we suppose that the operation is defined, then it leads the following contrariety:

$$\left(x\ \delta(x)\right)p(\frac{1}{x}) = 0 \quad \text{and} \quad \delta(x)\left(x\,p(\frac{1}{x})\right) = \delta(x),$$

where $\delta(x)$ is the Dirac distribution.

So in Schwart's theory, the expressions δ^2, δ^3,..............., δ^n are undefined.

To solve this problem, Colomboea *J.E.*[2] and his contemporaries studied the algebra of the objects referred to as "New Generalized Functions". After that Egorov[3] developed a simpler theory compared to" Colombo's" theory of generalized functions and defined it's applicability to nonlinear differential equations with partial derivatives.

The works of Antonevitch and Radyno[4] give a general construction method for the alegebras of new generalized functions and provide examples of its applications. Based on the Antonevitch -Radyno's approach, we published important results in this direction which have found applications in various fields of pure and applied mathematics[5-9].

In such algebras constructed[5-9], all the operations of multiplication, convolution, differentiation and the Fourier transformation are defined.

There arises a natural question : How is to define the Laplace transform in those algebras ?

The algebra of New Generalized functions $\xi(\prod(R))$ has been constructed;

so that

$$\prod^{/}(R) \subset S^{/}(R) \subset \xi(S(R)) \subset \xi(\prod(R))$$

where $\xi(S(R))$ - the space of New Generalized functions constructed in[5]

Preliminaries

We use the conventional notations

S - the space of test functions of rapid decay ;

L - the Laplace transform;

F - the Fourier transform;

*- the convolution;

S'- the space of tempered distributions.

We also use the definitions and some results[5]. Let us repeat some of them which are used throughout this study.

By $T(E)$ we denote the set of all possible sequences in E, where E be separated locally -convex algebra with topology defined by family of semi norms $(P_\alpha)_{\alpha\in A}$ such that for $\alpha \in A$, there exist $\beta \in A$ a constant $C_\alpha > o$ for which

$$\rho_\alpha(\lambda.\gamma) \le C_\alpha\, P_\beta(\lambda) P_\beta(\gamma) \quad \forall\ \lambda\ ,\ \gamma \in E \qquad (1)$$

Let $T^*(E)$ be the set of all sequences $\left(\lambda_k\right)_{k=n}^{\infty} \in E$ satisfy the following conditions there is a number m such that for each $\alpha \in A$, there is a nonnegative $\chi_\alpha > 0$ such that $P_\alpha(\lambda_k) \le \chi_\alpha\, k^m$ for each k. And $I^*(E)$ be the set of all sequences $\left(\lambda_k\right)_{k=n}^{\infty} \in E$ satisfy the following conditions for each number m and for each $\alpha \in A$, there is a nonnegative $\chi_\alpha > 0$ such that $P_\alpha(\lambda_k) \le \chi_\alpha\, k^{-m}$ for each k. The following results are true[5]:

Theorem 1

a. Let E be an algebra satisfies (1) then $T^*(E)$ is a sub algebra of algebra $T(E)$ and $I^*(E)$ is an Ideal in $T^*(E)$.

Corresponding Author: Dr. Sabra Ramadan, The Kingdom of Saudi Arabia, Ministry of Education, Jazan Teachers College, Department of Mathematics, Jazan -Abu Areesh,P.O. Box 203

b. All Spaces $S(R)$, $D(R)$, and $\xi(R)$ with their natural topology satisfy the inequality(1).

In 1993 we defined the space $\zeta(E)$ as a factor space $T^*(E)/I^*(E)$[5] and we proved many Important results for the space $\zeta(S(R))$. Also[8] we have defined the extended Fourier Transform $\overline{F}$: $\zeta(S(R)) \to \zeta(S(R))$.

The space $\zeta(\prod(R))$

Define the space $\prod(R) = \prod_1(R) \cup \prod_3(R)$ where

$\prod_1 = \left\{ \eta(t) \in C^\infty(R) : \lim_{t\to\infty} t^n \eta^{(k)}\ (t) = 0\ ,\ \forall\ n\ ,\ k \in Z \right\}$

$\prod_2 = \left\{ \eta(t) \in C^\infty[\,0\,,\infty) : \lim_{t\to\infty} t^n \eta^{(k)}\ (t) = 0\ ,\ \forall\ n\ ,\ k \in Z \right\}$

$\prod_3 = \left\{\ g(t) = \eta(|t|) :\ \eta(t) \in \prod_2(R) \right\}$.

We define topology on $\prod(R)$ by the following semi-norms

$P_\alpha(\eta(t)) = P_{n,l}(\eta(t)) = \sup_{k\le n,\, m\le l} q_{k,m}(\eta(t))$

where $q_{k,m}(\eta(t)) = \sup_{t\in(0,\infty)} \left| t^k \eta^{(m)}(t) \right|$

The space $(\prod(R)\,,\,P_\alpha)$ satisfies (1). So we conclude that $T^*(\prod(R))$ is a sub algebra of $T(\prod(R))$ and $I^*(\prod(R))$ be an Ideal in $T^*(\prod(R))$.

Moreover it easy to check the following results:

1. $S(R) \subset \prod(R)$, $T(S(R)) \subset T(\prod(R))$;
2. $\prod^{/}(R) \subset S^{/}(R)$, $\prod^{/}(R) \subset \zeta(S(R))$;
3. $T^*(S(R)) \subset T^*(\prod(R))$, $I^*(S(R)) \subset I^*(\prod(R))$.

The embedding of algebra $\zeta(S(R))$ in to the algebra $\zeta(\prod(R))$ is defined by the following mapping:

$$J_\pi : (\lambda_k) + I^*(S(R)) \to (\lambda_k) + I^*(\prod(R))$$

since if λ, $\gamma \in \zeta(S(R))$ and $J_\pi(\lambda) = J_\pi(\gamma)$, then $\lambda = (\lambda_k) + I^*(S(R))$ and $\gamma = (\gamma_k) + I^*(S(R))$, and $(\lambda_k - \gamma_k) \in I^*(\prod(R))$, but.

(λ_k), $(\gamma_k) \in T(S(R)))$

So we get the following results:

$\prod^{/}(R) \subset S^{/}(R) \subset \zeta(S(R)) \subset \zeta(\prod(R))$.

In algebra $\zeta(\prod(R))$ we define the associative multiplication for $\lambda = (\lambda_k) + I^*(S(R))$,

$\gamma = (\gamma_k) + I^*(S(R))$ by $\lambda\,\Theta\,\gamma = (\lambda_k\gamma_k) + I^*(\prod(R))$.

Theorem: The operation of multiplication Θ is independent of a representative.

Proof: Let $(\lambda_k^{/})$ and $(\gamma_k^{/})$ are any two other representative for λ and γ (respectively). Consider

$p_\alpha(\lambda_k\gamma_k - \lambda_k^{/}\gamma_k^{/}) \le p_\alpha(\lambda_k\gamma_k - \lambda_k\gamma_k^{/}) + p_\alpha(\lambda_k\gamma_k^{/} - \lambda_k^{/}\gamma_k^{/}) \le$

$C_{\alpha1}P_{\beta1}(\lambda_k)\ P_{\beta2}(\gamma_k - \gamma_k^{/}) + C_{\alpha2}P_{\beta3}(\gamma_k^{/})P_{\beta4}(\gamma_k - \gamma_k^{/}) \le C_\alpha k^{-m}$

.

Which means $\lambda_k\gamma_k \cong \lambda_k^{/}\gamma_k^{/}$.

So In algebra $\zeta(\prod(R))$ we can define the associative multiplication of distributions from $\prod^{/}(R)$ and $S^{/}(R)$.

Example: Let $\delta(x) \in S^{/}(R)$, then $\delta_\varphi^2 = (4\pi^2)^{-1}F^2(\varphi_k) + I(S(R))$.

For each $\psi \in S(R)$ we have

$$\left\langle F^2\varphi_k\,,\,\psi \right\rangle = \int_R k^2F^2(\varphi(kx))\varphi(x)dx = k\int_R F^2(\varphi(\tau))\,\psi(\frac{\tau}{k})d\tau =$$

$$= k\,C_{0\varphi}\langle\delta,\psi\rangle + C_{1\varphi}\langle\delta^{/},\psi\rangle + \frac{1}{k}C_{2\varphi}\langle\delta^{//},\psi\rangle + \ldots\ldots.. + \frac{1}{k^{n-1}}C_{n\varphi}\langle\delta^{(n)},\psi\rangle + \ldots\ldots$$

That is $\delta^2 = \dfrac{1}{4\pi^2}\sum C_{n\varphi}\dfrac{\delta^{(n)}}{k^{n-1}}$, where

$C_{n\varphi} = \int_R F^2(\varphi(\tau))d\tau$.

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