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# On Fuzzy Primary Γ-Ideals in Γ-Left Almost Semigroups

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**Abstract:** The purpose of this paper is to introduce the notion of a weakly fuzzy quasi-primary Γ-ideals in Γ-left almost semigroups, we study product of fuzzy primary, fuzzy quasi-primary, fuzzy weakly completely primary, weakly fuzzy primary and weakly fuzzy quasi-primary Γ-ideals in Γ-left almost semigroups. Moreover, we investigate relationships between fuzzy weakly completely primary and weakly fuzzy quasi-primary Γ-ideals in Γ-left almost semigroups.

**Keywords:** Fuzzy Primary, Fuzzy Quasi-Primary, Fuzzy Weakly Completely Primary, Weakly Fuzzy Primary, Weakly Fuzzy Quasi-Primary

## Introduction

A left almost semigroup is a generalization of semigroup with wide range of usages in theory of flocks (Naseeruddin, 1970). The fundamentals of this non associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). A non empty subset groupoid S is called a left almost semigroup if it satisfies the left invertive:

$$(ax)y = (yx)a$$

for all $a,x,y \in S$. It is interesting to note that a left almost semigroup with right identity becomes a commutative monoid (Mushtaq and Yousuf, 1978). Because of containing a right identity, a left almost semigroup becomes a commutative monoid (Mushtaq and Yousuf, 1978). A left identity in a left almost semigroup is unique (Mushtaq and Yousuf, 1978). It lies between a groupoid and a commutative semigroup with wide range of applications in theory of flocks (Naseeruddin, 1970). Ideals in left almost semigroups have been discussed in (Mushtaq and Yousuf, 1988). Now we define the concepts that we will used. Let $S$ be a left almost semigroup. By a left almost subsemigroup of (Mushtaq and Khan, 2009), we means a non-empty subset $B$ of $S$ such that $BB \subseteq B$. A non-empty subset $A$ of a left almost semigroup $S$ is called a right (left) ideal of $S$ (Mushtaq and Khan, 2007) if $AS \subseteq A(SA \subseteq A)$. By ideal or simply two-sided ideal, we mean a non-empty subset of a left almost semigrou $S$ which is both a right and a left ideal of $S$. In the notion of Γ-semigroups was introduced by (Sen, 1981). A groupoid is called a Γ-left almost semigroup if it satisfies the left invertive:

$$(a\alpha b)\beta x = (x\alpha b)\beta a$$

for all $a,b,x \in S$ and $\alpha,\beta \in \Gamma$ (Shah and Rehman, 2013). This structure is also known as an Γ-Abel-Grassmann's groupoid (Γ-AG-groupoid). In this study we are going to investigate some interesting properties of recently discovered classes, namely Γ-left almost semi group always satisfies the Γ-medial law:

$$(a\alpha b)\beta(x\gamma y) = (a\alpha x)\beta(b\gamma y)$$

for all $a,b,x,y \in S$ and $\alpha,\beta,\gamma \in \Gamma$ (Shah and Rehman, 2013), while a Γ-left almost semigroup with left (rigth) identity always satisfies Γ-paramedial:

$$(a\alpha b)\beta(x\gamma y) = (y\gamma x)\beta(b\alpha a)$$

for all $a,b,x,y \in S$ and $a,\beta,\gamma, \in \Gamma$ (Shah and Rehman, 2013). Recently Shah and Rehman have discussed Γ-bi-ideals and Γ-ideals in Γ-left almost semigroups. An Γ-ideal $P$ of a Γ-left almost semigroup $S$ is called primary if $X\Gamma Y \subseteq P$ implies that $X \subseteq P$ or $Y^n \subseteq P$ where Γ-ideals $A$ and $B$ in Mushtaq and Khan (2008) defined the direct product of right (resp, left) ideals, prime ideals, maximal ideals.

The fundamental concept of fuzzy subsets was first introduced by (Zadeh, 1965). Given a set $S'$, a fuzzy subset of $S'$ is, by definition a mapping $f$: $S' \rightarrow [0, 1]$, where [0,1] is the unit interval. Kuroki (1993) initiated the theory of fuzzy bi-ideals in semigroups. The thought of belongingness of a fuzzy point to a fuzzy set under a natural equivalence on a fuzzy $s$ set

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was defined by (Murali, 2004). Recently, Khan *et al*. (2010) introduced the concept of anti fuzzy ideals and fuzzy ideals of left almost semigroups in his papers (Khan *et al*., 2010). There are many mathematicians who added several results to the theory fuzzy Γ-left almost semigroups (Abdullaha *et al*., 2012). In this study we characterize the fuzzy subset in Γ-left almost semigroup. We investigate the relationships between fuzzy weakly completely primary and weakly fuzzy quasi-primary Γ-ideals in Γ-left almost semigroups.

## Preliminaries

Let $S$ be a Γ-left almost semigroup. A non empty subset $A$ of $S$ is called a left Γ-ideal of $S$ if $S\Gamma A\subseteq A$. A is called a right Γ-ideal of $S$ if $A\Gamma S\subseteq A$ and $A$ is called an Γ-ideal of $S$ if $A$ is both a left and a right Γ-ideal of $S$. A mapping $f$ from $S$ to the unit interval [0,1] is a fuzzy subset of $S$. The Γ-left almost semigroup $S$ itself is a fuzzy subset of $S$ such that $S(x) = 1$ for all $x\in S$, denoted also by $S$. Let $a$ and $b$ be two fuzzy subsets of $S$. Then the inclusion relation $a\subseteq b$ is defined $a(x)\leq b(x)$ for all $x\in S$. $a\cap b$ and $a\cup b$ are fuzzy subsets of $S$ defined by:

$$(a\cap b)(x) = min\{a(x), b(x)\}$$

$(a\cup b)(x)$ = max $\{a(x), b(x)\}$, for all $x\in S$. The product $a\Gamma b$ (Khan *et al*., 2013) is defined as follows:

$$a\Gamma b(z) = \begin{cases} \sup\left[\min\{a(x), b(y)\}\right]; z = x\gamma y \\ 0; z \neq x\gamma y. \end{cases}$$

As is well known (Khan *et al*., 2013). Fuzzy subset $a$ of $S$ is called a fuzzy sub Γ-left almost semigroup (fuzzy sub Γ-left almost semigroup) of $S$ if:

$$a(x\gamma y) \geq min\{a(x), a(y^n)\}$$

($a(x\gamma y)\geq\min\{a(x),a(y)\}$), for all $x,y$ in $S$, $\gamma\in\Gamma$ and is called a fuzzy left (right) Γ-ideal of S if $a(x\gamma y)\geq a(y)$ ($a(x\gamma y)\geq a(x)$) for all $\gamma\in\Gamma$, $x,y\in S$ if $a$ is both fuzzy right and left Γ-ideal of $S$, then $a$ is called a fuzzy Γ-ideal of $S$ (Khan *et al*., 2013). It is easy to see that $a$ is a fuzzy Γ-ideal of $S$ if and only if $a(x\gamma y)\geq\max\{a(x),a(y)\}$ for all $x,y\in S,\gamma\in\Gamma$ and any fuzzy right (left) Γ-ideal of S is a fuzzy Γ-left almost subsemigroup of $S$. Equivalently, We can prove easily that $A$ is a (right, left) Γ-ideal of $S$ if and only if the function $f_A$ of $A$ is a fuzzy (right, left) Γ-ideal of $S$ (Shah *et al*., 2014).

*Lemma 2.1.*

Khan *et al*. (2013) Let $S$ be a Γ-left almost semigroup. If $f$, $g$ and $h$ are fuzzy subsets of $S$, then $(f\Gamma g)\,\Gamma h = (h\Gamma g)\Gamma f$.

*Lemma 2.2.*

Khan *et al*. (2013) Let $S$ be a Γ-left almost semigroup with left identity. If $f,g,h$ and $k$ are fuzzy subsets of $S$, then:

- $f\Gamma(f\Gamma h) = g\Gamma(f\Gamma h)$;
- $(f\Gamma g)\,\Gamma(h\Gamma k) = (k\Gamma h)\Gamma\,(g\Gamma f)$.

*Lemma 2.3.*

Khan *et al*. (2013) Let $S$ be a Γ-left almost semigroup and $a$ be a fuzzy subset of $S$. Then the following properties hold.

- $a$ is a fuzzy right Γ-ideal of $S$ if and only if $a\Gamma S\subseteq a$.
- $a$ is a fuzzy left Γ-ideal of $S$ if and only if $S\Gamma a\subseteq a$.
- $a$ is a fuzzy Γ-ideal of $S$ if and only if $a\Gamma S\subseteq a$ and $S\Gamma a\subseteq a$.
- $a$ is a fuzzy sub Γ-left almost semigroup of $S$ if and only if $a\Gamma a\subseteq a$.

*Definition 2.4.*

Let $S$ be a Γ-left almost semigroup. A fuzzy subset $f$ of a Γ-left almost semigroup $S$ is called fuzzy quasi-primary if for any two fuzzy left Γ-ideals $g$ and $h$ of $S$ such that $g\Gamma h\subseteq f$ implies $g\subseteq f$ or $h^n\subseteq f$, for some positive integer $n$.

*Lemma 2.5.*

Khan *et al*. (2013) Let $S$ be a Γ-left almost semigroup and $f$ be a fuzzy left Γ-ideal of $S$. Then

- $S\Gamma S = S$;
- $S\Gamma f = f$.

*Definition 2.6.*

Let $S$ be a Γ-left almost semigroup. A fuzzy subset $f$ of $S$ is called fuzzy primary of $S$ if for any two fuzzy Γ-ideals $g$ and $h$ of $S$ such that $g\Gamma h\subseteq f$ implies $g\subseteq f$ or $h^n\subseteq f$, for some positive integer $n$.

*Remark*

Let $S$ be a Γ-left almost semigroup. It is easy to see that every fuzzy quasi-primary Γ-ideal is fuzzy primary.

*Definition 2.7.*

Let $S$ be a Γ-left almost semigroup. A fuzzy subset $f$ of $S$ is called fuzzy weakly completely primary if $\max\{f(x), f(y^n)\}\geq f(x\gamma y)$, for some positive integer $n$, where $x,y\in S$ and $\gamma\in\Gamma$.

*Theorem 2.8.*

Let $S$ be a Γ-left almost semigroup. If $f$ is a fuzzy weakly sub Γ-left almost semigroup of $S$, then 1-$f$ is fuzzy weakly completely primary.

*Proof.*

Assume that $f$ is a fuzzy weakly sub Γ-left almost semigroup of $S$. Since

$$f(x\gamma y) \geq min\{f(x), f(y^n)\},$$

we have $1-f(x\gamma y) \leq 1-\max\{f(x), f(y^n)\}$, for some positive integer $n$, where $x,y \in S$ and $\gamma \in \Gamma$. If $f(x) \leq f(y^n)$, then $1-f(x) \geq 1-f(y^n)$, for all positive integer $n$. Then:

$$\begin{aligned} max\{1-f(x), 1-f(y^n)\} &= 1-f(x) \\ &= 1-min\{f(x), f(y^n)\} \\ &\leq 1-f(x\gamma y) \end{aligned}$$

so that 1-$f$ is fuzzy weakly completely primary. If $f(x) > f(y^n)$ for some positive integer $n$, we have the same result. Thus

$$\begin{aligned} max\{1-f(x), 1-f(y^n)\} &= 1-f(y^n) \\ &= 1-min\{f(x), f(y^n)\} \\ &\leq 1-f(x\gamma y) \end{aligned}$$

so that 1-$f$ is fuzzy weakly completely primary.

*Theorem 2.9.*

Let $S$ be a Γ-left almost semigroup. If 1-$f$ is fuzzy weakly completely primary left Γ-ideal of $S$, then $f$ is fuzzy sub Γ-left almost semigroup of $S$.

*Proof.*

Suppose that 1-$f$ is fuzzy weakly completely primary of $S$. Since

$$max\{1-f(x), 1-f(y^n)\} \geq 1-f(x\gamma y),$$

we have

$$1-max\{1-f(x), 1-f(y^n)\} \leq f(x\gamma y)$$

for some positive integer $n$, where $x,y \in S$ and $\gamma \in \Gamma$. Thus $f(x\gamma y) \geq \max\{f(x), f(y^n)\} \geq \min\{f(x), f(y)\}$ and hence $f$ is a fuzzy sub Γ-left almost semigroup of $S$.

*Theorem 2.10.*

Let S be a Γ-left almost semigroup. If $P_i, i \in I$ are fuzzy weakly completely primary subsets of S, then $\bigcup_{i\in I} P_i$ is fuzzy weakly completely primary subset of $S$.

*Proof.*

Suppose that $P_i, i \in I$ are fuzzy weakly completely primary subset of $S$. Then

$$P_i(x\gamma y) \leq max\{P_i(x), P_i(y^n)\}$$

for some positive integer $n$, where $x,y \in S$, $\gamma \in \Gamma$ and for $i \in I$. Since

$$max\left\{\bigcup_{i\in I} P_i(x), \bigcup_{i\in I} P_i(y^n)\right\} \geq P_i(x\gamma y),$$

for all $i \in I$, we get

$$max\left\{\bigcup_{i\in I} P_i(x), \bigcup_{i\in I} P_i(y^n)\right\} \geq \bigcup_{i\in I} P_i(x\gamma y).$$

Hence $\bigcup_{i\in I} P_i$ is fuzzy weakly completely primary subset of $S$.

*Theorem 2.11.*

Shah *et al*. (2014) Let $I$ be a non-empty subset of a Γ-left almost semigroup $S$ and $f_I$: $S \rightarrow [0,1]$ be a fuzzy subset of $S$ such that:

$$f_I(x) = \begin{cases} 1; x \in I \\ 0; x \notin I \end{cases}$$

Then $I$ is a left Γ-ideal (right Γ-ideal, Γ-ideal) of $S$ if and only if $f_I$ is a fuzzy left Γ-ideal (resp. fuzzy right Γ-ideal, fuzzy Γ-ideal) of $S$.

*Theorem 2.12.*

Let $I$ be an Γ-ideal (left, right Γ-ideal) of a Γ-left almost semigroup $S, t \in (0,1]$. If $f_I$ is fuzzy set of $S$ such that:

$$f_I(x) = \begin{cases} t; x \in I \\ 0; x \notin I \end{cases}$$

Then $f_I$ is a fuzzy Γ-ideal (fuzzy left, fuzzy right Γ-ideal) of $S$.

*Definition 2.13.*

Shah *et al.* (2014) Let $S$ be a Γ-left almost semigroup, $x \in S$ and $t \in [0,1]$. A fuzzy point $x_t$ of $S$ is defined by the rule that:

$$x_t(y)=\begin{cases}t; x=y\\0; x\neq y\end{cases}$$

It is accepted that $x_t$ is a mapping from $S$ into [0,1], then a fuzzy point of $S$ is a fuzzy subset of $S$. For any fuzzy subset $f$ of $S$, we also denote $x_t \subseteq f$ by $x_t \in f$ in sequel. Let $tf_A$ be a fuzzy subset of S defined as follows:

$$tf_A(x)=\begin{cases}t; x\in A\\0; x\notin A\end{cases}$$

*Lemma 2.14.*

Let $A$ be a subset of a $\Gamma$-left almost semigroup $S$ and $f$ be a fuzzy set of $S$. Then the following statements are equivalent

- $tg_A \subseteq f$, $t\in[0,1]$;
- $A \subseteq f_t$, $t\in[0,1]$.

*Definition 2.15.*

A fuzzy subset $f$ of $S$ is said to be a weakly fuzzy primary if $tg_A\Gamma th_B \subseteq f$ implies $tg_A \subseteq f$ or $th_B^n \subseteq f$, for some positive integer $n$, where $A$ and $B$ are two $\Gamma$-ideals of $S$ and $t\in(0,1]$.

*Definition 2.16.*

A fuzzy subset $f$ of $S$ is said to be a weakly fuzzy quasi-primary if $tg_A\Gamma th_B \subseteq f$ implies $tg_A \subseteq f$ or $th_B^n \subseteq f$, for some positive integer n, where $A$ and $B$ are two left $\Gamma$-ideals of $S$ and $t\in(0,1]$.

*Remark*

It is easy to see that every weakly fuzzy quasi-primary is weakly fuzzy primary.

## Fuzzy Quasi-Primary Γ-Ideals of Γ-Left Almost Semigroups

The results of the following lemmas seem to play an important role to study fuzzy primary ideals in Γ- left almost semigroups; these facts will be used frequently and normally we shall make no reference to this lemma.

*Lemma 3.1.*

Let $A$,$B$ be any non-empty subset of a $\Gamma$-left almost semigroup $S$. Then for any $t\in(0,1]$ the following statements are true:

1. $tf_A\Gamma tf_B = tf_{A\Gamma B}$
2. $tf_A \cap tf_B = tf_{A\cap B}$
3. $tf_A \cup tf_B = tf_{A\cup B}$
4. $tf_A = \bigcup_{a\in A} a_t$
5. $S\Gamma tf_A = tf_{S\Gamma A}, tf_A\Gamma S = tf_{A\Gamma S}$ and $S\Gamma(tf_A\Gamma S) = tf_{S\Gamma(A\Gamma S)}$
6. If $A$ is a left $\Gamma$-ideal (right, $\Gamma$-ideal) of $S$, then $tf_A$ is a fuzzy left $\Gamma$-ideal (fuzzy left, fuzzy $\Gamma$-ideal) of $S$

*Proof.*

Proof is straightforward.

*Theorem 3.2.*

Let $P$ be an $\Gamma$-ideal of $\Gamma$-left almost semigroup $S$. Then $P$ is a primary $\Gamma$-ideal of $S$ if and only if the fuzzy subset $f_P$ is a fuzzy weakly completely primary $\Gamma$-ideals of $S$.

*Proof.*

(⇒) Suppose that $P$ is a primary $\Gamma$-ideal of $S$. Obviously, $f_P$ is a fuzzy subset of S. Let $x,y\in S$ and $\gamma\in\Gamma$. If $x\gamma y\in P$, then

$$f_P(x\gamma y)=0\leq \max\{f_P(x), f_P(y^n)\}$$

for some positive integer $n$. Let $x\gamma y\in P$. Then $f_P(x\gamma y)$ = 1. Since $P$ is a primary $\Gamma$-ideal of $S$, we have $x\in P$ or $y^n\in P$ for some positive integer $n$. Thus $f_P(x) = 1$ or $f_P(y)$ = 1 and so $f_P(x\gamma y)$ = 1= max $\{f_P(x), f_P(y)\}$. Therefore the fuzzy subset P is a fuzzy weakly completely primary $\Gamma$-ideals of $S$.

(⇐) Suppose that $f_P$ is a fuzzy weakly completely primary $\Gamma$-ideals of $S$. Let $x,y\in S$ be such that $x\gamma y\in P$. Then $f_P(x\gamma y)$ = 1. Since $f_P$ is a fuzzy weakly completely primary $\Gamma$-ideals of $S$, we have $1=f_P(x\gamma y) \leq \max\{f_P(x), f_P(y^n)\}$ for some positive integer $n$. Thus $f_P(x)=1$ or $f_P(y^n)=1$ and so $x\in P$ or $y^n\in P$, for some positive integer $n$. Therefore $P$ is a primary $\Gamma$-ideal of $S$.

*Theorem 3.3.*

Let $f$ be a fuzzy subsets of $\Gamma$-left almost semigroups $S$. Then $f$ is a fuzzy weakly completely primary $\Gamma$-ideal of $S$ if and only if the level subset $f_t, t\in \mathrm{Im}(f)$ of $f$ is a weakly completely primary $\Gamma$-ideal of $S$, for every $t\in[0,1]$.

*Proof.*

(⇒) Suppose that $f$ is a fuzzy weakly completely primary $\Gamma$-ideal of $S$. Let $x,y\in S,\gamma\in\Gamma$ such that $x\gamma y\in f_t$. Then $f(x\gamma y)\geq t$. Since $f$ is a fuzzy weakly completely primary $\Gamma$-ideal of $S$, we have $f(x\gamma y)\leq$ max $\{f(x), f(y^n)\}$, for some positive integer $n$. If $f(x)\leq f(y^n)$, then $t\leq\max\{f(x)\leq f(y^n)\}=f(x)$ and $f(x)\geq t$, so $x\in f_t$. If $f(x) > f(y^n)$, then $t\leq\max\{f(x), f(y^n)\}=f(y^n)$ and $f(y^n)\geq t$, so $y^n\in f_t$.

(⇐) Suppose that $f_t$ is a weakly completely primary $\Gamma$-ideal of $S$, for every $t\in[0,1]$. Let $x,y\in S$ and $\gamma\in\Gamma$. Then $f(x\gamma y)\geq 0$. Since $x\gamma y\in f_{f(x\gamma y)}$, by hypothesis, we have $x\in f_{f(x\gamma y)}$ or $y^n\in f_{f(x\gamma y)}$, for some positive integer $n$. Thus $f(x)\geq f(x\gamma y)$ or $f(y^n)\geq f(x\gamma y)$ and hence $\max\{f(x), f(y^n)\}\geq f(x\gamma y)$.

*Theorem 3.4.*

Let $P$ be a fuzzy left $\Gamma$-ideal of a $\Gamma$-left almost semigroup with left identity $S$. Then the following statements are equivalent:

1. $P$ is a weakly fuzzy quasi-primary of $S$.
2. For any $x,y\in S$ and $t\in(0,1]$, if $x_t\Gamma(S\Gamma y_t)\subseteq P$, then $x_t\in P$ or $y_t^n\in P$, for some positive integer $n$.
3. For any $x,y\in S$ and $t\in(0,1]$, if $tf_x\Gamma tf_y\subseteq P$, then $x_t\in P$ or $y_t^n\in P$, for some positive integer $n$.
4. If $A$ and $B$ are left $\Gamma$-ideals of $S$ such that $tf_A\Gamma tf_B\subseteq P$, then $tf_A\subseteq P$ or $tf_B^n\subseteq P$, for some positive integer $n$.

*Proof.*

($1\Rightarrow 2$) Let $P$ be a weakly fuzzy quasi-primary of $S$. For any $x,y\in S$ and $t\in(0,1]$, if $x_t\Gamma(S\Gamma y_t)\subseteq P$, then

$$\begin{aligned}tf_{(xe)S}\Gamma tf_{(ye)S} &= (tf_{(xe)}\Gamma S)\Gamma(tf_{ye}\Gamma S)\\ &= (tf_{(xe)}\Gamma tf_{ye})\Gamma(S\Gamma S)\\ &= ((tf_x\Gamma tf_e)\Gamma(tf_y\Gamma tf_e))\Gamma(S\Gamma S)\\ &= ((tf_x\Gamma tf_y)\Gamma(tf_e\Gamma tf_e))\Gamma(S\Gamma S)\\ &= ((tf_e\Gamma tf_e)\Gamma(tf_y\Gamma tf_x))\Gamma(S\Gamma S)\\ &= (tf_{ee}\Gamma(tf_y\Gamma tf_x))\Gamma(S\Gamma S)\\ &= (tf_y\Gamma(tf_e\Gamma tf_x))\Gamma(S\Gamma S)\\ &= (tf_y\Gamma tf_{ex})\Gamma(S\Gamma S)\\ &= (S\Gamma S)\Gamma(tf_x\Gamma tf_y)\\ &= S\Gamma(tf_x\Gamma tf_y)\\ &= tf_x\Gamma(S\Gamma tf_y)\\ &= x_t\Gamma(S\Gamma y_t)\\ &\subseteq P.\end{aligned}$$

Since $P$ is a weakly fuzzy quasi-primary $\Gamma$-ideal, we get

$$x_t = tf_x = tf_{(e\gamma e)\alpha x} = tf_{(x\gamma e)\alpha e}\subseteq tf_{(x\gamma e)\Gamma S}\subseteq P$$

or

$$y_t^n = tf_{y^n} = tf_{((e\gamma e)\alpha y)^n} = tf_{((y\gamma e)\alpha e)^n}\subseteq tf_{((y\gamma e)\Gamma S)^n}\subseteq P,$$

for some positive integer $n$. Hence $x_t\in P$ or $y_t^n\in P$, for some positive integer $n$.

($2\Rightarrow 3$) Let $x,y\in S, t\in(0,1]$ and:

$$tf_x\Gamma tf_y\subseteq P.$$

Then

$$\begin{aligned}x_t\Gamma(S\Gamma y_t) &\subseteq tf_x\Gamma(S\Gamma tf_y\\ &= S\Gamma(tf_x\Gamma tf_y)\\ &\subseteq S\Gamma P\\ &\subseteq P.\end{aligned}$$

Thus, by hypothesis $x_t\in P$ or $y_t^n\in P$, for some positive integer $n$.

($3\Rightarrow 4$) Let $A$ and $B$ be left $\Gamma$-ideals of $S$. Then, by Lemma 3.1, we get $tf_A$ and $tf_B$ are fuzzy left $\Gamma$-ideals of $S$. Suppose that $tf_A\Gamma tf_B\subseteq P$ and $tf_{B^n}\not\subset P$, then there exists $y\in B$ such that $y_t^n\notin P$, for all positive integer $n$. For any $x\in A$ and $\gamma\in\Gamma$, by Lemma 3.1 and hypothesis:

$$\begin{aligned}tf_x\Gamma tf_y &= tf_{x\gamma y}\\ &\subseteq tf_{A\Gamma B}\\ &= tf_A\Gamma tf_B\\ &\subseteq P.\end{aligned}$$

Since $y_t^n\notin P, tf_{y^n}\nsubseteq P$, which implies $tf_x\subseteq P$ and so $x_t\in P$. By Lemma 3.1, it follows that $tf_A=\bigcup_{x\in A}x_t$.

($4\Rightarrow 1$) Let $A$ and $B$ are left $\Gamma$-ideals of $S$ such that $tf_A\Gamma tf_B\subseteq P$. Thus, by hypothesis $tf_A\subseteq P$ or $tf_B^n\subseteq P$, for some positive integer $n$. By Definition 2.16, we get $P$ is a weakly fuzzy quasi-primary of $S$.

*Corollary 3.5.*

Let $P$ be a fuzzy $\Gamma$-ideal of a $\Gamma$-left almost semigroup with left identity $S$. Then the following statements are equivalent:

1. $P$ is a weakly fuzzy primary $\Gamma$-ideal of $S$.
2. For any $x,y\in S$ and $t\in(0,1]$, if $x_t\Gamma(S\Gamma y_t)\subseteq P$, then $x_t\in P$ or $y_t^n\in P$ for some positive integer $n$.
3. For any $x,y\in S$ and $t\in(0,1]$, if $tf_x\Gamma tf_y\subseteq P$, then $x_t\in P$ or $y_t^n\in P$, for some positive integer $n$.
4. If $A$ and $B$ are $\Gamma$-ideals of $S$ such that $tf_A\Gamma tf_B\subseteq P$, then $tf_A\subseteq P$ or $tf_B^n\subseteq P$ for some positive integer $n$.

*Proof.*

It is straightforward by Theorem 3.4.

*Theorem 3.6.*

Let $S$ be a $\Gamma$-left almost semigroup with left identity. If $f$ is a fuzzy quasi-primary of $S$, then $inf\{f(a^2\Gamma(S\Gamma b^2))\}\leq max\{f(a^2),f(b^2)^n\}$, for some positive integer $n$, where $a,b\in S$.

*Proof.*

Suppose that $f$ is a fuzzy quasi-primary Γ-ideal of $S$ and $inf\{f(a^2\Gamma(S\Gamma b^2))\} > max\{f(a^2), f(b^2)^n)\}$. Let $inf\{f(a^2\Gamma(S\Gamma b^2))\} = m.$ Define two fuzzy subsets $g$ and $h$ of $S$ as follows:

$$g(x) = \begin{cases} m; x \in a^2\Gamma S \\ 0; x \notin a^2\Gamma S \end{cases}$$

and:

$$h(x) = \begin{cases} m; x \in b^2\Gamma S \\ 0; x \notin b^2\Gamma S \end{cases}$$

Then $g$ and h are fuzzy left Γ-ideals of $S$ by theorem 2.12. If:

$$g\Gamma h(x) = sup[min\{g(y), h(z)\}] = m$$

Then there exists an $u\in a^2\Gamma S$, $v\in b^2\Gamma S$ such that $u\gamma v = x$. Put $u = a^2\alpha t$ and $v = b^2\beta k$, for some $t,k\in S$ and $\gamma,\alpha,\beta\in\Gamma$. Then

$$\begin{aligned} f(x) &= f(u\gamma v) \\ &= f((a^2\alpha t)\gamma(b^2\beta k)) \\ &= f((a^2\alpha b^2)(t\beta k)) \\ &= f((k\beta t)\gamma(b^2\alpha a^2)) \\ &\geq f(b^2\alpha a^2) \\ &= f(a^2\alpha b^2) \\ &= f(a^2\alpha(e\delta b^2)) \\ &\geq inf(f(a^2\Gamma(S\Gamma b^2))) \\ &= m. \end{aligned}$$

So that $g\Gamma h\subseteq f$. Since $f$ is a quasi-primary Γ-ideal, we get $g \subseteq f$ or $h^n \subseteq f$, for some positive integer $n$. Thus

$$\begin{aligned} h^n(x) &= \bigcup_{x=a_1\gamma_1 b_1} min\{h^{n-1}(a_1), h(b_1)\} \\ &= \bigcup_{x=a_1\gamma_1 b_1} min\left\{\bigcup_{a_1=a_2\gamma_2 b_2} min\{h^{n-2}(a_2), h(b_2)\}, h(b_1)\right\} \\ &\vdots \\ &= \bigcup_{x=(a_n\gamma_n b_n)\gamma_{n-1}b_{n-1}\ldots b_1} min\{min\{h(a_n), h(b_n)\},\ldots, h(b_1)\} \end{aligned}$$

and $(b^2)^n = (b^2b^2)b^2\ldots b^2$. Then $g(a^2) = g(a^2e) = m$ or

$$\begin{aligned} h^n((b^2)^n) &= \bigcup_{x=(b^2\gamma b^2)\gamma b^2\ldots b^2} min\{min\{h(b^2), h(b^2)\},\ldots, h(b^2)\} \\ &= \bigcup_{x=(b^2\gamma b^2)\gamma b^2\ldots b^2} min\left\{\begin{matrix} min\{h((ee)b^2), h((ee)b^2)\},\ldots, \\ h((ee)b^2) \end{matrix}\right\} \\ &= \bigcup_{x=(b^2\gamma b^2)\gamma b^2\ldots b^2} min\{min\{h(b^2e), h(b^2e)\},\ldots, h(b^2e)\} \\ &= \bigcup_{x=(b^2\gamma b^2)\gamma b^2\ldots b^2} min\{min\{m, m\},\ldots, m\} \\ &= m. \end{aligned}$$

But from $m = max\{f(a^2), f(b^2)^n\} < inf\{f(a^2(Sb^2))\} = m$ we have a contradiction.

*Theorem 3.7.*

Let $S$ be a Γ-left almost semigroup with left identity and $P$ is a fuzzy Γ-ideal of $S$. If $P(x\gamma y)=max\{P(x),P(y^n)\}$, then $P$ is a weakly fuzzy quasi-primary Γ-ideal of $S$, for some positive integer $n$, where $x,y\in S$ and $\gamma\in\Gamma$.

*Proof.*

Suppose that $x_t,y_t(t\in(0,1])$ are the fuzzy points of $S$ such that $x_t\Gamma(S\Gamma\, y_t)\subseteq P$. Since

$$\begin{aligned} S\Gamma(x\gamma y)_t &= S\Gamma(x_t\gamma y_t) \\ &\subseteq x_t\Gamma(S\Gamma y_t) \\ &\subseteq P \end{aligned}$$

and $P(x\gamma y)= max\{P(x),P(y^n)\}$, we have $P(x\gamma y)\geq t$, which implies that $P(x)\geq t$ *or* $P(y^n)\geq t$, for some positive integer $n$. Then $x_t\in P$ or $y_t^n \in P$.

*Corollary 3.8.*

Let $S$ be a Γ-left almost semigroup with left identity. If $P$ is a fuzzy weakly completely primary, then $P$ is weakly fuzzy quasi-primary of $S$.

*Proof.*

It is straightforward by Theorem 3.7.

*Theorem 3.9.*

Let $S$ be a Γ-left almost semigroup with left identity. A fuzzy subset $P$ of a Γ-left almost semigroup $S$ is weakly fuzzy quasi-primary if and only if $P(x\gamma y)\leq max\{P(x),P(y^n)\}$, for some positive integer $n$, where $x,y\in S$ and $\gamma\in\Gamma$.

*Proof.*

Suppose that $P$ is a weakly fuzzy quasi-primary of $S$. If $P(x\gamma y)> max\{P(x),P(y^n)\}$, then there exists $t\in(0,1)$ such that

$$P(x\gamma y) > t > max\left\{P(x), P(y^n)\right\}.$$

Thus

$$\begin{aligned} x_t\gamma(S\gamma y_t) &= S\gamma(x_t\gamma y_t) \\ &= S\gamma(x\gamma y)_t \\ &\subseteq S\Gamma P \\ &\subseteq P \end{aligned}$$

for all $x,y\in S$ and $\gamma\in\Gamma$. Since $P$ is a weakly fuzzy quasi-primary of S, we get $x_t\in P$ or $y_t^n \in P$, for some positive integer $n$, but $x_t\notin P$ and $y_t^n \notin P$, which is impossible. Therefore

$$P(x\gamma y) \le max\left\{P(x), P(y^n\right\}$$

for all $x,y\in S$ and $\gamma\in\Gamma$.

## Product of Fuzzy Γ-Ideals of Γ-Left Almost Semigroups

We start with the following theorem that gives a relation between product of fuzzy Γ-ideal and fuzzy Γ-ideal in Γ-left almost semigroup.

Our starting points are the following definitions. Let $S_1$ and $S_2$ be two Γ-left almost semigroups. Then

$$S_1\times S_2 := \left\{(x,y)\in S_1\times S_2 \mid x\in S_1, y\in S_2\right\}$$

and for any $(a,b),(c,d)\in S_1\times S_2,\gamma\in\Gamma$ we define $(a,b)\gamma(c,d):=(a\gamma c,b\gamma d)$, then $S_1\times S_2$ is a Γ-left almost semigroup as well. Let $f:S_1\rightarrow[0,1]$ and $g:S_2\rightarrow[0,1]$ be two fuzzy subsets of Γ-left almost semigroups $S_1$ and $S_2$ respectively. Then the product of fuzzy subsets is denoted by $f\times g$ and defined as $f\times g:S_1\times S_2\rightarrow[0,1]$, where:

$$(f\times g)(x,y) = min\left\{f(x), g(y)\right\}$$

*Lemma 4.1.*

If $f$ and $g$ are fuzzy sub Γ-left almost semigroups of $S_1$ and $S_2$ respectively, then $f\times g$ is a fuzzy subΓ-left almost semigroup of $S_1\times S_2$.

*Proof.*

Proof is straightforward.

*Lemma 4.2*

If $f$ and $g$ are fuzzy left Γ-ideals (fuzzy right Γ-ideals, fuzzy Γ-ideals) of $S_1$ and $S_2$ respectively, then $f\times g$ is a fuzzy left Γ-ideal (fuzzy right Γ-ideal, fuzzy Γ-ideal) of $S_1\times S_2$.

*Proof.*

Proof is straightforward.

*Corollary 4.3.*

Let $f_1, f_2, f_3,\ldots, f_n$ be a fuzzy subsets of Γ-left almost simigroups $S_1, S_2, S_3,\ldots, S_n$ respectively:

1. If $f_1, f_2, f_3,\ldots, f_n$ are fuzzy subΓ-left almost semigroups of $S_1, S_2, S_3,\ldots, S_n$ respectively, then $\prod_{i=1}^{n} f_i$ is fuzzy subΓ-left almost semigroup of $\prod_{i=1}^{n} S_i$.
2. If $f_1, f_2, f_3,\ldots, f_n$ are fuzzy left Γ-ideals (fuzzy right Γ-ideals, fuzzy Γ-ideals) of $S_1, S_2, S_3,\ldots, S_n$ respectively, then $\prod_{i=1}^{n} f_i$ is fuzzy left Γ-ideal (fuzzy right Γ-ideal, fuzzy Γ-ideal) of $\prod_{i=1}^{n} S_i$.

*Proof.*

By mathematical induction.

*Lemma 4.4.*

Let $f$, $g$ be two fuzzy subsets of Γ-left almost simigroups $S_1$, $S_2$ respectively and $t\in[0,1]$. Then $(f\times g)_t=f_t\times g_t$.

*Proof*

Proof is straightforward.

*Corollary 4.5.*

Let $f_1,f_2,f_3,\ldots, f_n$ are fuzzy subsets of Γ-left almost simigroups $S_1,S_2,S_3,\ldots,S_n$ respectively and and $t\in[0,1]$. Then $(\prod_{i=1}^{n} f_i)_t = \prod_{i=1}^{n} (f_i)_t$.

*Proof*

By mathematical induction.

*Theorem 4.6.*

Let $f$ and $g$ be two fuzzy weakly completely primary (fuzzy primary, quasi-primary) Γ-ideals of a Γ-left almost semigroups $S_1$, $S_2$ respectively. Then $(f\times g)$ is a fuzzy weakly completely primary (fuzzy primary, quasi-primary) Γ-ideal of $S_1\times S_2$.

*Proof*

Let $(a,b),(c,d)\in S_1\times S_2$ and $\gamma\in\Gamma$. Since $f$ and $g$ are fuzzy weakly completely primary Γ-ideals of $S$, we get:

$$\begin{aligned}(f\times g)((a,b)\gamma(c,d)) &= (f\times g)(a\gamma c, b\gamma d)\\ &= \min\{f(a\gamma c), g(b\gamma d)\}\\ &\le \min\{\max\{f(a), f(c^n)\}, \max\{g(b), g(d^n)\}\}\\ &= \max\{\min\{f(a), g(b^n)\}, \min\{f(c^n), g(d)\}\}\\ &\le \max\{\min\{f(a), g(b)\}, \min\{f(c), g(d)\}\}\\ &= \max\{(f\times g)(a,b), (f\times g)(c,d)\}\end{aligned}$$

for some positive integer $n$. Hence $f\times g$ is a fuzzy weakly completely primary of $S_1\times S_2$.

*Theorem 4.7.*

Let $f_1$, $f_2$ be a fuzzy subsets of Γ-left almost semigroups $S_1$, $S_2$ respectively. Then $f\times g$ is a fuzzy weakly completely primary Γ-ideal of $S_1\times S_2$ if and only if the level subset $(f\times g)_t$, $t\in Im(f\times g)$ of $f\times g$ is a weakly completely primary Γ-ideal of $S_1\times S_2$, for every $t\in[0,1]$.

*Proof*

(⇒) Suppose that $f\times g$ is a fuzzy weakly completely primary Γ-ideal of $S_1\times S_2$. Let $(x,y),(m,n)\in S_1\times S_2, \gamma\in\Gamma$ such that:

$$(x,y)\gamma(m,n)\in(f\times g)_t$$

Then:

$$(f\times g)((x,y)\gamma(m,n))\ge t.$$

So that $(f\times g)(x\gamma m, y\gamma m)\ge t$. Since $f\times g$ is a fuzzy weakly completely primary Γ-ideal of $S_1\times S_2$, we have:

$$(f\times g)((x,y)\gamma(m,n))\le \max\{(f\times g)(x,y), (f\times g)(m,n)^k\}$$

for some positive integer $k$. If $(f\times g)(x,y)\le(f\times g)(m,n)^k$, then:

$$t\le \max\{(f\times g)(x,y), (f\times g)(m,n)^k\} = (f\times g)(m,n)^k$$

and $(f\times g)(m,n)^k \ge$t, so $(m,n)^k\in(f\times g)_t$. If $(f\times g)(x,y)> (f\times g)(m,n)^k$, then:

$$\le \max\{(f\times g)(x,y), (f\times g)(m,n)^k\} = (f\times g)(x,y)$$

and $(f\times g)(x,y),\ge t$, so $(x,y)\in(f\times g)_t$.

(⇐) Suppose that $(f\times g)_t$ is a weakly completely primary Γ-ideal of $S_1\times S_2$, for every $t\in[0,1]$. Let $(x,y)$, $(m,n)\in S_1\times S_2$ and $\gamma\in\Gamma$. Then:

$$(f\times g)((x,y)\gamma(m,n))\ge 0\ .$$

Since $(x,y)\gamma(m,n)\in(f\times g)_{(f\times g)((x,y)\gamma(m,n))}$, by hypothesis, we have $(x,y)\in(f\times g)_{(f\times g)((x,y)\gamma(m,n))}$ or $(m,n)^k\in(f\times g)_{(f\times g)((x,y)\gamma(m,n))}$, for some positive integer $k$. Thus:

$$(f\times g)(x,y)\ge(f\times g)((x,y)\gamma(m,n))$$

or

$$(f\times g)(m,n)^k\ge(f\times g)((x,y)\gamma(m,n))$$

and hence:

$$\max\{(f\times g)(x,y), (f\times g)(m,n)^k\}\ge(f\times g)((x,y)\gamma(m,n)).$$

*Corollary 4.12.*

Let $f_1,f_2,f_3,\dots,f_n$ be a fuzzy subsets of Γ-left almost semigroups $S_1,S_2,S_3,\dots,S_n$ respectively and and $t\in[0,1]$. Then $\prod_{i=1}^{n} f_i$ is a fuzzy weakly completely primary Γ-ideal of $\prod_{i=1}^{n} S_i$ if and only if the level subset $(\prod_{i=1}^{n} f_i)_t, t\in Im(\prod_{i=1}^{n} S_i)$ is a weakly completely primary Γ-ideal of $\prod_{i=1}^{n} S_i$.

*Proof*

By mathematical induction.

## Conclusion

Many new classes of Γ-left almost semigroups have been discovered recently. All these have attracted researchers of the field to investigate these newly discovered classes in detail. This article investigates the weakly fuzzy quasi-primary Γ-ideals in Γ-left almost semigroups. Some characterizations of product of fuzzy primary, fuzzy quasi-primary, fuzzy weakly completely primary, weakly fuzzy primary and weakly fuzzy quasi-primary Γ-ideals in Γ-left almost semigroups. Finally, we obtain necessary and sufficient conditions of a fuzzy weakly completely primary and weakly fuzzy quasi-primary Γ-ideals in Γ-left almost semigroups

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