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A Generalization of the Modified Liouville Equation

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Abstract: We study the modified Liouville equation using various transformations to build dynamical systems and we use Dulac's criterion for give sufficient conditions of the non-existence of periodic orbits in the dynamical systems generated of the modified Liouville equation.

Keywords: Modified Liouville Equation, Dulac Functions, Bendixson-Dulac Criterion, Periodic Orbits

Introduction

The Bendixson-Dulac criterion consists of a sufficient number of conditions for the nonexistence of periodic orbits in planar dynamical systems (Farkas, 1994). The modified Liouville equation (Abdelrahman *et al*., 2015; Salam *et al*., 2012) plays an important role in various areas of mathematical physics, from plasma physics and field theoretical modeling to fluid dynamics, using various transformations the differential equation can be written as a dynamic system that under some conditions does not have periodic orbits (Marin *et al*., 2014; 2013a; Osuna and Villaseñor, 2011). The system in (Marin-Ramirez *et al*., 2015) coincides to our system. A generalization of a dynamical system was made in (Yan-Min *et al*., 2016; Qiu-Peng *et al*., 2015; Xiangwei *et al*., 2016). A Dulac function for a quadratic system was found in (Marin *et al*., 2013b). A Dulac function and a geometric method for a quadratic system was studied in (Marin-Ramirez *et al*., 2014). In this article our objective is construct dynamical systems that does not have periodic orbits using Dulac functions and we use the following criterion to show the non-existence of periodic orbits. The Dulac criterion was used in (Rana, 2015).

Theorem 1.1 (Bendixson-Dulac criterion) Let $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ and $h(x_1, x_2)$ be functions C^1 in a simply connected domain $D \subset \mathbb{R}^2$ such that $\frac{\partial(hf_1)}{\partial x_1} + \frac{\partial(hf_2)}{\partial x_2}$ does not change sign in D and vanishes at most on a set of measure zero. Then the system:

$$\begin{aligned} x_1' &= f_1(x_1, x_2) \\ x_2' &= f_2(x_1, x_2), \ (x_1, x_2) \in D \end{aligned} \tag{1}$$

Does not have periodic orbits in D.

We need to find a function $h(x_1, x_2)$, which satisfies the conditions of the theorem of Bendixson-Dulac, that is called a Dulac function.

Preliminary Results

Techniques to Construction of Dulac Functions

Definition 2.1 Let $C^0(D, \mathbb{R})$ be the set of continuous functions and define $\Omega = \{f \in C^0(D, \mathbb{R}) : f$ does not change sign and vanishes only on a measure zero set$\}$.

Theorem 2.2 If there exist $c(x_1, x_2) \in \Omega$ such that h is a solution of the system:

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h\left(c(x_1, x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right) \tag{2}$$

with $h \in \Omega$, then for Equation 1 h is a Dulac function on D. (Osuna and Villaseñor, 2011).

The Modified Liouville Equation

$$a^2 u_{xx} - u_{tt} + be^{\beta u} = 0 \tag{3}$$

where, a, b and β are non zero and arbitrary coefficients.

Using the wave transformation $u(x, t) = u(\xi)$ $\xi = kx+wt$ with:

$$v = e^{\beta u}$$
$$u = \frac{\ln(v)}{\beta}, \ u_{xx} = \frac{k^2}{\beta} \frac{v''v - v'^2}{v^2} \tag{4}$$

and:

$$u_{tt} = \frac{w^2}{\beta}\frac{v''v - v'^2}{v^2}$$

Equation 3 can be reduced to:

$$\delta v''v - \delta v'^2 + bv^3 = 0 \tag{5}$$

where, $\delta = \frac{k^2a^2}{\beta} - \frac{w^2}{\beta}$ Taking $\mu(v) = v'(\xi)$ and $v''(\xi) = \mu(v)\mu'(v)$ then:

$$bv^3 - \delta\mu(v)^2 + \delta v\mu'(v)\mu(v) = 0$$

We obtain:

$$\frac{2b}{\delta}v^2 - \frac{2}{v}\theta + \theta' = 0$$

where, $\theta = \mu^2$. Multiplying in both sides by v^{-2} we get $\frac{2b}{\delta} + (v^{-2}\theta)' = 0$. Integrating with respect to v:

$$\frac{2b}{\delta}v + C_1 + v^{-2}\theta = 0$$

But $\theta = \mu^2$ and also $\mu(v) = v'(\xi)$. Hence:

$$v'(\xi) = \pm\sqrt{-\frac{2b}{\delta}v^3 + C_1v^2}$$

As $v'(\xi) = dv/d\xi$ we obtain:

$$\int dv \Big/ \sqrt{-\frac{2b}{\delta}v^3 + C_1v^2} = \pm\xi + C_2$$

If $C_1 = 0$ then $\frac{i\sqrt{\frac{2\delta}{b}}}{\sqrt{v}} = \pm\xi + C_2$. In consequence $v = -\frac{2\delta}{b(\pm\xi + C_2)^2}$.

and:

$$u = \frac{1}{\beta}\ln\frac{-2\delta}{b(\pm\xi + C_2)^2} \tag{6}$$

If $C_1 \neq 0$ with the substitution $w = \sqrt{-C_1 + \frac{2bv}{\delta}}$ such that $v = \frac{\delta}{2b}\left(w^2 + C_1\right)$, then:

$$-i\int\frac{dv}{v}\frac{1}{w} = -i\int\frac{\delta}{2b}\frac{2wdw}{\frac{\delta}{2b}(w^2 + C_1)}\frac{1}{w} = -2i\int\frac{dw}{w^2 + C_1}$$

and:

$$-2i\frac{\tan^{-1}\left(\frac{\sqrt{-C_1 + \frac{2bv}{\delta}}}{\sqrt{C_1}}\right)}{\sqrt{C_1}} = \pm\xi + C_2$$

or:

$$\frac{\sqrt{-C_1 + \frac{2bv}{\delta}}}{\sqrt{C_1}} = \tan\left(i\frac{\sqrt{C_1}}{2}\left(\pm\xi + C_2\right)\right)$$
$$= i\tanh\left(\frac{\sqrt{C_1}}{2}\left(\pm\xi + C_2\right)\right)$$

It follows that:

$$-C_1 + \frac{2bv}{\delta} = -C_1\tanh^2\left(\frac{\sqrt{C_1}}{2}\left(\xi \pm C_2\right)\right)$$

Hence:

$$v(\xi) = \frac{\delta}{2b}\left[C_1 - C_1\tanh^2\left(\frac{1}{2}\sqrt{C_1}\left(\xi \pm C_2\right)\right)\right]$$

If $\delta c_1 = C$ and $c_2 = B$ then the general solution of this differential equation is:

$$v(\xi) = \frac{C}{2b}sech^2\left[\frac{\sqrt{C}}{2\sqrt{\frac{k^2a^2 - w^2}{\beta}}}(\xi + B)\right]$$

where, C and B are constants, $k^2a^2 - w^2 \neq 0$.

From Equation 4 and $C = 2b$ then:

$$u = \frac{1}{\beta}\ln sech^2\left[\frac{\sqrt{2b}}{2\sqrt{\frac{k^2a^2 - w^2}{\beta}}}(\xi + B)\right] \tag{7}$$

From Equation 3 and $u(x, t) = u(\xi)$ we obtain:

$$\left(a^2k^2 - w^2\right)u'' + be^{\beta u} = 0 \tag{8}$$

Integrating and taking the constant of integration equal to 0:

$$(a^2k^2 - w^2)\frac{u'^2}{2} + \frac{be^{\beta u}}{\beta} = 0$$

Integrating the last equation with respect to ξ and taking u, $u' \to 0$ when $\xi \to \pm\infty$, we get the constant of integration $\frac{b}{\beta}$ in the solution given in Equation 7.

Dynamical System

From Equation 5 and making a change of variables:

$$v' = x_2, v = x_1$$

$$\delta x'_2 x_1 - \delta x_2^2 + b x_1^3 = 0$$

with $\delta = \frac{k^2 a^2}{\beta} - \frac{w^2}{\beta}$. We obtain the following system:

$$\begin{cases} x_1' = x_2 \\ x_2' = \frac{x_2^2}{x_1} - \frac{b}{\delta} x_1^2 \end{cases} \tag{9}$$

with $x_1 \neq 0$. Let us show that the previous dynamical system does not have periodic orbits. From Equation 2:

$$x_2 \frac{\partial h}{\partial x_1} + \left(\frac{x_2^2}{x_1} - \frac{b x_1^2}{\delta} \right) \frac{\partial h}{\partial x_2} = h \left(c - \frac{2x_2}{x_1} \right) \tag{10}$$

Supposing that $\frac{\partial h}{\partial x_1} = 0$, $\frac{\partial h}{\partial x_2} = h$, $h = e^{x_2}$ then Equation 10 becomes:

$$c(x_1, x_2) = \frac{x_2^2}{x_1} + \frac{2x_2}{x_1} - \frac{b x_1^2}{\delta} \tag{11}$$

where, $c(x_1, x_2) < 0$ for b, $\delta > 0$ then some of the plane regions are:

$$D_1 = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 < 3\sqrt{\frac{-\delta}{b}} \right\}$$

$$D_2 = \left\{ \left(x_1, x_2\right) \in \mathbb{R}^2 : 3\sqrt{\frac{-\delta}{b}} < x_1 < 0, x_2 < -\sqrt{1 + \frac{b}{\delta} x_1^3} - 1 \right\}$$

$$D_2 = \left\{ \left(x_1, x_2\right) \in \mathbb{R}^2 : 3\sqrt{\frac{-\delta}{b}} < x_1 < 0, x_2 > \sqrt{1 + \frac{b}{\delta} x_1^3} - 1 \right\}$$

$$D_4 = \left\{ x_1 > 0, -\sqrt{1 + \frac{b}{\delta} x_1^3} - 1 < x_2 < \sqrt{1 + \frac{b}{\delta} x_1^3} - 1 \right\}$$

Main Results

Theorem 4.1 The system of Equation 9 can be generalized as:

$$\begin{cases} \dot{x}_1 = c_2(x_2) \\ \dot{x}_2 = \frac{x_2^2}{x_1} + c_1(x_1) e^{-x_2} - \frac{b x_1^2}{\delta} \end{cases}$$

and does not have periodic orbits at simply connected domains $D_{1,2,3,4} \subset \mathbb{R}^2$.

Proof. Replacing Equation 11 and $h = e^{x_2}$ with their derivatives into Equation 2:

$$f_2 + \frac{\partial f_2}{\partial x_2} = \frac{x_2^2}{x_1} + \frac{2x_2}{x_1} - \frac{b x_1^2}{\delta}$$

Solving the previous differential equation by integrating factor, we have:

$$f_2 = \frac{x_2^2}{x_1} + c_1(x_1) e^{-x_2} - \frac{b x_1^2}{\delta}$$

Then, from $\frac{\partial f_1}{\partial x_1} = 0$, $f_1 = c_2(x_2)$ and we have proved the theorem.

Example 4.2 If we consider that c_2 has a first derivative and it is invertible such that $c'_2\left(c_2^{-1}(z)\right)$ exists for all z in which $c_2^{-1}(z)$ is defined, then we have the generalized modified Liouville equation:

$$x_1 \ddot{x}_1 = c'_2\left(c_2^{-1}(\dot{x}_1) \right) \left(\left(c_2^{-1}(\dot{x}_1) \right)^2 + c_1(x_1) e^{-c_2^{-1}(\dot{x}_1)} x_1 - \frac{b x_1^3}{\delta} \right)$$

If $c_1(x_1) = 0$ and $c_2(x_2) = x_2$ we have the modified Liouville equation

The parametrization $\frac{d\xi}{d\tau} = x_1$ transforms the system of Equation 9 into:

$$\dot{x}_2 x_1 = \frac{dx_2}{d\xi} x_1 = \frac{dx_2}{d\xi} \frac{d\xi}{d\tau} = \frac{dx_2}{d\tau} = x_2^2 - \frac{b}{\delta} x_1^3$$

$$\frac{dx_1}{d\xi} = \frac{dx_1}{d\tau} \frac{d\tau}{d\xi} = \frac{dx_1}{d\tau} \frac{1}{\frac{d\xi}{d\tau}} = x_2$$

Then $\frac{dx_1}{d\tau} = \frac{d\xi}{d\tau} x_2 = x_1 x_2$ we have an equivalent system:

$$\begin{cases} \dot{x}_1 = x_1 x_2 \\ \dot{x}_2 = x_2^2 - \frac{b}{\delta} x_1^3 \end{cases}$$

Theorem 4.3 The system of Equation 9 can be generalized to:

$$\begin{cases} \dot{x}_1 = x_1 x_2 + c_2(x_2) \\ \dot{x}_2 = x_2^2 + c_1(x_1) e^{-x_2} - \frac{b x_1^3}{\delta} \end{cases}$$

and does not have periodic orbits at simply connected domain in $\mathbb{R}^2$.

Proof. Replacing $C(x_1, x_2) = x_2^2 + 3x_2 - \frac{b}{\delta}x_1^3$ with $9 + 4\frac{b}{\delta}x_1^3 < 0$ in Equation 2, we obtain:

$$f_2 + \frac{\partial f_2}{\partial x_2} = x_2^2 + 2x_2 - \frac{bx_1^3}{\delta}$$

Solving the previous differential equation, we have:

$$f_2 = x_2^2 + c_1(x_1)e^{-x_2} - \frac{bx_1^3}{\delta}$$

Then, from $\frac{\partial f_1}{\partial x_1} = x_2$, $f_1 = x_1x_2 + c_2(x_2)$ and we have proved the theorem.

Conclusion

Several solutions were obtained taking different values of the constant of integration. The corresponding system of the modified Liouville equation was generalized. Using travelling waves, the modified Liouville equation was transformed into a dynamical system and, with the use of Dulac's criterion, we gave sufficient conditions for the nonexistence of periodic orbits in four domains. By differentiable transformations other dynamical systems can be obtained first set of equations. Here, we can get a new generalization of this system. These results are important for the study of nonlinear partial differential equations. Very interesting future work is the generalization of the original partial differential equation to the modified Liouville equation in time and space. Also, we can consider a family of Dulac functions $h = \exp(ax_2)$ for different values of the parameter a. In this study, we worked with $a = 1$.

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Author's Contributions

All Authors have contributed to the research and writing of the paper.

Ethics

The authors declare that there is no conflict of interests regarding the publication of this paper.

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