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Application of an Enhanced (G'/G)-Expansion Method to Find Exact Solutions of Nonlinear PDEs in Particle Physics

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Abstract: The enhanced (G'/G)-expansion method is very effective and powerful method to find the exact traveling wave solutions of nonlinear evolution equations. We choose the Phi-4 equation to illustrate the validity and advantages of this method. As a result, many exact traveling wave solutions are obtained, which include soliton, hyperbolic function and trigonometric function solutions.

Keywords: Enhanced (G'/G)-Expansion, Phi-4 Equation, Traveling Wave Solutions, Solitons, Nonlinear Evolution Equations

Introduction

In recent year, the exact traveling wave solution for nonlinear Partial Differential Equations (PDEs) has been investigated by various authors who are nonlinear substantial phenomena. Many powerful method have been obtainable for instance the exp(-Φ(ξ))-expansion method (Khan *et al*., 2013a; Islam *et al*., 2014); the jacobi elliptic function method (Ali, 2011); the homogeneous balance method (Wang, 1995; Zayed *et al*., 2004); the modified simple equation method (Jawad *et al*., 2010; Khan and Akbar, 2013b; Zayed and Ibrahim, 2012; Akter and Akbar, 2015); the (G′/G)-expansion method (Wang *et al*., 2008; Zayed, 2010; Akbar *et al*., 2012b; Zayed and Gepreel, 2009; Akbar and Ali, 2011; Shehata, 2010; Akbar *et al*., 2012a; Mirzazadeh *et al*., 2014; Alam and Akbar, 2014a); the improve (G′/G)-expansion method (Zhang *et al*., 2010); the extended(G'/G)-expansion method (Roshid *et al*., 2014a; 2014b; Alam and Akbar, 2014b); the generalized (G'/G)-expansion method (Alam *et al*., 2014a; 2014b; 2014c); the novel (G'/G)-expansion method (Hafez *et al*., 2014); the homotopy perturbation method (Mohyud-Din *et al*., 2011a; 2011b; 2011c); the variational method (He, 1997; Abbasbandy, 2007; Arife and Yildirim, 2011; Abdou and Soliman, 2005); the exp-function method (Akbar and Ali, 2012; He and Wu, 2006; Naher *et al*., 2012); the truncated painleve expansion method (Weiss *et al*., 1983); the asymptotic method (He, 2008); the Hirota's bilinear transformation method (Hirota, 1973; Hirota and Satsuma, 1981); the tanh-function method (Abdou, 2007; Fan, 2000; Malfliet, 1992); the F-expansion method (Wang and Li, 2005); the generalized Riccati equation (Yan and Zhang, 2001); the ansatz method (Sassaman and Biswas, 2009a; 2010; Sassaman *et al*., 2010a; 2010b; Chowdhury and Biswas, 2012); the perturbation method (Biswas *et al*., 2008; Sassaman and Biswas, 2009b; Biswas *et al*., 2012a); the lie symmetry method (Biswas *et al*., 2013); the method of integrability (Biswas *et al*., 2012b) and so on.

The objective of this article is to bring to bear the enhanced (G'/G)-expansion method to extract new exact traveling wave solutions and then solitary wave solutions to the Phi-4 equation. This application shows the simplicity, efficiency and effectiveness of the enhanced (G'/G)-expansion method. To the best of our knowledge the enhanced (G′/G)-expansion method has not been applied to the above mentioned equation in the previous literature.

The article is organized as follows: In section 2, we have discussed the description of the method and its application. In section 3, the advantages of the method, comparison, physical explanation and graphical representation of the obtained solutions have been discussed. Finally, in section 4, we have drawn our conclusions.

Materials and Methods

In this section, we discuss the enhanced (G'/G)-expansion method to yields some new and more general exact traveling wave solutions of the Phi-4 equation.

Description of the Enhanced (G′/G)-Expansion Method

In this sub section, we describe in details the enhanced (G'/G)-expansion method for finding traveling wave solutions of nonlinear equations. Any nonlinear equation in two independent variables x and t can be expressed in following form:

$$\Psi(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}.........) = 0 \tag{2.1}$$

where, $u(\xi) = u(x, t)$ is an unknown function, Ψ is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and non linear terms are involved. The following steps are involved in finding the solution of nonlinear Equation (2.1) using this method.

Step 1: The given PDE (2.1) can be transformed into ODE using the transformation $\xi = x \pm \omega t$, where ω is the speed of traveling wave such that $\omega \in R$-{0}.

The traveling wave transformation permits us to reduce Equation 2.1 to the following ODE:

$$\Psi(u, u', u'', ...) = 0 \tag{2.2}$$

where, Ψ is a polynomial in $u(\xi)$ and its derivatives, where $u'(\xi) = \frac{du}{d\xi}$, $u''(\xi) = \frac{d^2u}{d\xi^2}$, and so on.

Step 2: Now we suppose that the Equation (2.2) has a general solution of the form:

$$u(\xi) = \sum_{i=-n}^{n} \left(\frac{a_i (G'/G)^i}{(1 + \lambda (G'/G))^i} + b_i (G'/G)^{i-1} \sqrt{\sigma \left(1 + \frac{(G'/G)^2}{\mu} \right)} \right) \tag{2.3}$$

Subject to the condition that $G = G(\xi)$ satisfy the equation:

$$G'' + \lambda G = 0 \tag{2.4}$$

where, a_i, b_i ($-n \le i \le n$; $n \in N$) and λ are constant to be determined, provided that $\sigma = \pm 1$ and $\mu \ne 0$.

Step 3: The positive integer n can be determined by balancing the highest order derivatives to the highest order nonlinear terms appear in Equation 2.1 or in Equation 2.2. More precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = n$ which gives rise to the degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q,\ D\left(u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n + q) \tag{2.5}$$

Step 4: We substitute Equation 2.3 into Equation 2.2 and use Equation 2.4. We then collect all the coefficient of $(G'/G)^j$ and $(G'/G)^j \sqrt{\sigma \left(1 + \frac{(G'/G)^2}{\mu} \right)}$ together. Since Equation 2.3 is a solution of Equation 2.2. we can set each of the coefficient equal to zero which leads to a system of algebraic equations in terms of a_i, b_i ($-n \le i \le n$; $n \in N$), λ and ω. One can solves easily these system equations using Maple.

Step 5: For $\mu < 0$ general solution of Equation 2.4 gives:

$$\frac{G'}{G} = \sqrt{-\mu} \tanh\left(A + \sqrt{-\mu}\ \xi\right) \tag{2.6}$$

and:

$$\frac{G'}{G} = \sqrt{-\mu} \coth\left(A + \sqrt{-\mu}\ \xi\right) \tag{2.7}$$

and for $\mu > 0$, we get:

$$\frac{G'}{G} = \sqrt{\mu} \tan\left(A - \sqrt{\mu}\ \xi\right) \tag{2.8}$$

and:

$$\frac{G'}{G} = \sqrt{\mu} \cot\left(A + \sqrt{\mu}\ \xi\right) \tag{2.9}$$

where, A is an arbitrary constant. Finally we can construct a number of families of travelling wave solutions of Equation 2.1 by substituting the values of a_i, b_i ($-n \le i \le n$; $n \in N$), λ and ω (obtained in Step 3) and using Equation 2.6 to 2.9 into Equation 2.3.

Application of the Method

In this sub-section, the Phi-4 equation is a very important Nonlinear Evolution Equations (NLEEs) in the area of Mathematical Physics. The Phi-4 equation is considered as a particular form of the Klein-Gordon equation that model phenomenon in particle physics where kink and anti-kink solitary waves interact. The phi-4 equation is studied in various areas of Physics includes Plasma Physics, Fluid Dynamics, Quantum Field Theory, Solid State Physics and others (Ehsani *et al*., 2013). We will exploit the enhanced (G'/G)-expansion method to solve the phi-4 equation. Let us consider the phi-4 equation is in the form:

$$u_{tt} - u_{xx} + m^2 u + \lambda u^3 = 0 \tag{3.1}$$

where, m and λ are real valued constants, the terms u_{tt} and u_{xx} represents the effect of dissipation and the term u^3 represents the nonlinearity effect. Using the traveling wave variable $\xi = x - \omega t$, Equation 2.1 is transformed into the following ODE for $u = u(\xi)$:

$$(\omega^2 - 1)u'' + m^2 u + \lambda u^3 = 0 \quad (3.2)$$

where, primes denotes the differentiation with regard to ξ. By balancing u'' and u^3, we obtain $N = 1$. Therefore, the enhanced (G'/G)-expansion method admits to solution of (2.1) in the form:

$$u(\xi) = a_0 + \frac{a_1(G'/G)}{1+\lambda(G'/G)} + \frac{a_{-1}(1+\lambda(G'/G))}{(G'/G)} + b_0(G'/G)^{-1}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)} + b_1\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)} + b_{-1}(G'/G)^{-2}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)} \quad (3.3)$$

where, G = G (ξ) satisfies Equation 2.4. Substituting Equation 3.3 into Equation 3.2 and using Equation 2.4, we get a polynomial in $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$. Setting the coefficient of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$ equal to zero, we obtain a system containing a large number of algebraic equations in terms of unknown coefficients. We have solved this system of equations using Maple 13 and obtained the following set of solutions:

Set 1:

$$\omega = \pm\frac{\sqrt{-2\mu(m^2-2\mu)}}{2\mu},\ a_0 = \pm\sqrt{\lambda\mu}\,m,$$
$$a_1 = \mp\frac{m(1+\lambda^2\mu)}{\sqrt{\lambda\mu}},\ a_{-1} = 0,\ b_0 = 0,\ b_1 = 0,\ b_{-1} = 0$$

Set 2:

$$\omega = \pm\frac{\sqrt{\mu(m^2+\mu)}}{\mu},\ a_0 = 0,\ a_1 = 0,$$
$$a_{-1} = 0,\ b_0 = 0,\ b_1 = \pm\frac{2m}{\sqrt{-2\lambda\sigma}},\ b_{-1} = 0$$

Set 3:

$$\omega = \pm\frac{\sqrt{\mu(m^2+\mu)}}{\mu},\ a_0 = 0,\ a_1 = 0,\ a_{-1} = 0,\ b_0 = \pm\sqrt{\frac{-2\mu}{\lambda\sigma}},\ b_1 = 0,\ b_{-1} = 0$$

Set 4:

$$\omega = \pm\frac{\sqrt{-2\mu(m^2-2\mu)}}{2\mu},\ a_0 = \mp m\sqrt{\lambda\mu},\ a_1 = 0,\ a_{-1} = \pm\frac{\sqrt{\lambda\mu}\,m}{\lambda},\ b_0 = 0,\ b_1 = 0,\ b_{-1} = 0$$

Set 5:

$$\omega = \pm\frac{\sqrt{-\mu(2m^2-\mu)}}{\mu},\ a_0 = \mp\sqrt{\lambda\mu}\,m,\ a_1 = 0,\ a_{-1} = \pm\frac{\sqrt{\lambda\mu}\,m}{\lambda},\ b_0 = \pm\frac{\sqrt{\mu}\,m}{\sqrt{\lambda\sigma}},\ b_1 = 0,\ b_{-1} = 0$$

Substituting Set 1-Set 5 into Equation 3.3 along with Equation 2.6-2.9; we get the following families of traveling wave solutions.

Hyperbolic function solutions: When $\mu<0$, we get the following five families of hyperbolic function solutions.

Family 1:

$$u_{1,2}(x,t) = \frac{\pm m}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu - \sqrt{-\mu}\tanh(A+\sqrt{-\mu}\,\xi)}{1+\lambda\sqrt{-\mu}\tanh(A+\sqrt{-\mu}\,\xi)}\right)$$
$$u_{3,4}(x,t) = \frac{\pm m}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu - \sqrt{-\mu}\coth(A+\sqrt{-\mu}\,\xi)}{1+\lambda\sqrt{-\mu}\coth(A+\sqrt{-\mu}\,\xi)}\right)$$

where, $\xi = x \pm \frac{\sqrt{-2\mu(m^2-2\mu)}\,t}{2\mu}$

Family 2:

$$u_{5,6}(x,t) = \frac{\mp 2mI}{\sqrt{2\lambda}}\operatorname{sec}h(A+\sqrt{-\mu}\,\xi)$$
$$u_{7,8}(x,t) = \frac{\pm 2m}{\sqrt{2\lambda}}\operatorname{csc}h(A+\sqrt{-\mu}\,\xi)$$

where, $\xi = x \pm \frac{\sqrt{\mu(m^2+\mu)}\,t}{\mu}$

Family 3:

$$u_{9,10}(x,t) = \frac{\pm 2m}{\sqrt{2\lambda}}\operatorname{csc}h(A+\sqrt{-\mu}\,\xi),$$
$$u_{11,12}(x,t) = \frac{\mp 2mI}{\sqrt{2\lambda}}\operatorname{sec}h(A+\sqrt{-\mu}\,\xi)$$

where, $\xi = x \pm \frac{\sqrt{\mu(m^2+\mu)}\,t}{\mu}$

Family 4:

$$u_{13,14}(x,t) = \frac{\mp I\,m}{\sqrt{\lambda}}\coth(A+\sqrt{-\mu}\,\xi),$$
$$u_{15,16}(x,t) = \frac{\pm I\,m}{\sqrt{\lambda}}\tanh(A+\sqrt{-\mu}\,\xi)$$

where, $\xi = x \pm \frac{\sqrt{-2\mu(m^2-2\mu)}\,t}{2\mu}$

Family 5:

$$u_{17,18}(x,t) = \frac{\mp I\,m}{\sqrt{\lambda}}(\coth(A+\sqrt{-\mu}\,\xi) - \operatorname{csc}h(A+\sqrt{-\mu}\,\xi)),$$
$$u_{19,20}(x,t) = \frac{\mp m}{\sqrt{\lambda}}(\operatorname{sec}h(A+\sqrt{-\mu}\,\xi) - I\tanh(A+\sqrt{-\mu}\,\xi))$$

where, $\xi = x \pm \frac{\sqrt{-\mu(2m^2-\mu)}\,t}{\mu}$

Trigonometric function solutions: When μ>0, we get the following five families of trigonometric function solutions.

Family 6:

$$u_{21,22}(x,t) = \frac{\pm m}{\sqrt{\lambda}}\left(\frac{\lambda\sqrt{\mu} - \tan(A-\sqrt{\mu}\,\xi)}{1+\lambda\sqrt{\mu}\tan(A-\sqrt{\mu}\,\xi)}\right),$$
$$u_{23,24}(x,t) = \frac{\pm m}{\sqrt{\lambda}}\left(\frac{\lambda\sqrt{\mu} - \cot(A+\sqrt{\mu}\,\xi)}{1+\lambda\sqrt{\mu}\cot(A+\sqrt{\mu}\,\xi)}\right)$$

where, $\xi = x \pm \frac{\sqrt{-2\mu(m^2-2\mu)}\,t}{2\mu}$

Family 7:

$$u_{25,26}(x,t) = \mp\frac{2mI}{\sqrt{2\lambda}}\sec(A-\sqrt{\mu}\,\xi),$$
$$u_{27,28}(x,t) = \mp\frac{2mI}{\sqrt{2\lambda}}\csc(A+\sqrt{\mu}\,\xi)$$

where, $\xi = x \pm \frac{\sqrt{\mu(m^2+\mu)}\,t}{\mu}$

Family 8:

$$u_{29,30}(x,t) = \pm\frac{2mI}{\sqrt{2\lambda}}\csc(A-\sqrt{\mu}\,\xi),$$
$$u_{31,32}(x,t) = \pm\frac{2mI}{\sqrt{2\lambda}}\sec(A+\sqrt{\mu}\,\xi)$$

where, $\xi = x \pm \frac{\sqrt{\mu(m^2+\mu)}\,t}{\mu}$,

Family 9:

$$u_{33,34}(x,t) = \pm\frac{m}{\sqrt{\lambda}}\cot(A-\sqrt{\mu}\,\xi),$$
$$u_{35,36}(x,t) = \pm\frac{m}{\sqrt{\lambda}}\tan(A+\sqrt{\mu}\,\xi)$$

where, $\xi = x \pm \frac{\sqrt{-2\mu(m^2-2\mu)}\,t}{2\mu}$,

Family 10:

$$u_{37,38}(x,t) = \pm\frac{m}{\sqrt{\lambda}}(\cot(A-\sqrt{\mu}\,\xi) + \csc(A-\sqrt{\mu}\,\xi)),$$
$$u_{39,40}(x,t) = \pm\frac{m}{\sqrt{\lambda}}(\tan(A+\sqrt{\mu}\,\xi) + \sec(A+\sqrt{\mu}\,\xi))$$

where, $\xi = x \pm \frac{\sqrt{-\mu(2m^2-\mu)}\,t}{\mu}$,

Remark: All the obtained solutions have been checked with maple by putting them back into the original equations and found correct. In Family 3 and 4, the solutions $u_{5,6}$ (x, t) and $u_{7,8}$ (x, t) are coincide with the solutions $u_{11,12}$ (x, t) and $u_{9,10}$ (x, t) respectively.

Discussion

In this section, we will discuss the advantages, comparison between (Akter and Akbar, 2015) solutions and our solutions, physical explanations and graphical representation of the above determined ten families of the solutions.

Advantages and Comparison

By means of the enhanced (G'/G)-expansion method, we have found forty solutions of the phi-4 equation. On the other hand, Akter and Akbar (2015) have found only four solutions of the phi-4 equation through the modified simple equation method to see below the Appendix.

Appendix

Akter and Akbar (2015) investigated solutions of the Phi-4 equation by the modified simple equation method and they obtained the following solutions:

$$u(x,t) = \pm\sqrt{-\frac{m^2}{\lambda}}\tanh\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}m(x-Vt)\right) \quad \text{(R.1)}$$

and:

$$u(x,t)=\pm\sqrt{-\frac{m^2}{\lambda}}\coth\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}m(x-Vt)\right) \quad \text{(R.2)}$$

Using hyperbolic function identities, Equation R.1 and R.2 can be written as:

$$u(x,t)=\mp i\sqrt{-\frac{m^2}{\lambda}}\tan\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}i\,m(x-Vt)\right) \quad \text{(R.3)}$$

and:

$$u(x,t)=\pm i\sqrt{-\frac{m^2}{\lambda}}\cot\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}i\,m(x-Vt)\right) \quad \text{(R.4)}$$

The main advantages of the enhanced (G'/G)-expansion method over the modified simple equation method is that it provides more new exact traveling wave solutions along with additional free parameters. Moreover, if we compare between these two methods and if we also focus on our newly generated solutions, the enhanced (G'/G)-expansion method is more effective in providing many new solutions than the modified simple equation method. The comparison between (Akter and Akbar, 2015) solutions and our solutions are given Table 1.

Beyond this table, all others solutions are new exact traveling wave solutions which are not being establish in the previous literature.

Physical Explanation

The introduction of dispersion without introducing nonlinearity destroys the solitary wave as different Fourier harmonics start propagating at different group velocities. On the other hand, introducing nonlinearity without dispersion also prevents the formulation of solitary waves, because the pulse energy is frequently pumped into higher frequency models. However, if both dispersion and nonlinearity are present, solitary waves can be sustained. Similarity to dispersion, dissipation can also give rise to solitary wave when combined with nonlinearity. Hence it is more interesting to point out that the delicate balance between the nonlinearity effect of u^3 and the dissipative effect of u_{xx} gives rise to solitons solitary waves, that after a full interaction with others the solitons come back retaining their identities with the same speed and shape. The Phi-4 equation has many solitary wave solutions. There is various type of traveling wave solutions that one of particular interest in solitary wave theory. The type of traveling wave depends on the variation of the physical parameters. If the exact solutions of the Phi-4 equation arise in a complex form according to the variations of the physical parameters, then the wave propagation for any varied instance is characterize by $|u(x, t)|$. For some special values of the physical parameters, the traveling wave solutions originate from the obtained exact explicit solutions as follows:

The solitary wave solutions of kink type corresponding to u_1 (x,t) for the fixed values of the parameters $m = 0.2$, $\mu = -0.5$, $A = 2$, $\lambda = 1$ within $-1 \le x \le 1$ and $0 \le t \le 1$ have presented in Fig. 1.

Table 1. Comparison between the new solutions and (Akter and Akbar, 2015) solution

(Akter and Akbar, 2015) Solution	Our solution
If $m=2, \lambda=4, \mathrm{V}=\sqrt{3}$,	If $m=2,\ \lambda=4,\ \mu=-1,\ \mathrm{A}=0$ and
Solution (from the Equation 3.36) becomes:	$u_{15,16}(x,t)=u(x,t)$, Solution $u_{15,16}(x,t)$ becomes:
$u(x,t)=\pm\ I\tanh(x-\sqrt{3}\,t)$	$u(x,t)=\pm\ I\tanh(x-\sqrt{3}\,t)$
If $m=2, \lambda=4, \mathrm{V}=\sqrt{3}$,	If $m=2,\ \lambda=4,\ \mu=-1,\ \mathrm{A}=0$ and
Solution (from the Equation 3.37) becomes:	$u_{13,14}(x,t)=u(x,t)$, Solution $u_{13,14}(x,t)$ becomes:
$u(x,t)=\pm\ I\coth(x-\sqrt{3}\,t)$.	$u(x,t)=\pm\ I\coth(x-\sqrt{3}\,t)$
If $m=-1, \lambda=1, \mathrm{V}=\frac{1}{\sqrt{2}}, i=\sqrt{-1}$,	If $m=-1,\ \lambda=1,\ \mu=1,\ \mathrm{A}=0$ and
Solution (from the Equation 3.38) becomes:	$u_{35,36}(x,t)=u(x,t)$, Solution $u_{35,36}(x,t)$ becomes:
$u(x,t)=\pm\tan(x-\frac{1}{\sqrt{2}}t)$	$u(x,t)=\pm\tan(x-\frac{1}{\sqrt{2}}t)$
If $m=-1, \lambda=1, \mathrm{V}=\frac{1}{\sqrt{2}}, i=\sqrt{-1}$,	If $m=-1,\ \lambda=1,\ \mu=1,\ \mathrm{A}=0$ and
Solution (from the Equation 3.39) becomes:	$u_{33,34}(x,t)=u(x,t)$, Solution $u_{33,34}(x,t)$ becomes:
$u(x,t)=\pm\cot(x-\frac{1}{\sqrt{2}}t)$	$u(x,t)=\pm\cot(x-\frac{1}{\sqrt{2}}t)$

Figure 2 shows the solitary wave solutions of singular kink type corresponding to u_9 (x,t) with fixed parameters m = -0.10, μ = -1.57, A = 1, λ = 1 within the interval -3 ≤ x, t ≤ 3. The bell type solitary wave solution corresponding to u_{11} (x,t) for the fixed values of the parameters m = 0.2, μ = -1.5, A = 0.5, λ = 1 and -3 ≤ x, t ≤ 3 is shown in Fig. 3. Again, for the values m = -0.15, μ = -1, A = 0, λ = 1.5 and -3 ≤ x, t ≤ 3, solution u_5 (x,t) are also given the exact solitary wave solutions of bell type. The bell type solitary wave solution is shown in Fig. 6. It has infinite wings or infinite tails. This soliton referred to as non topological solitons. This solution does not depend on the amplitude and high frequency soliton. Figure 4 shows the shape of exact solitary wave solution of singular soliton; obtained from the solution u_{17} (x,t) corresponding to the fixed values m = 4, μ = -0.8, A = 0, λ = 2, -3 ≤ x, t ≤ 3. The exact periodic traveling wave solutions corresponding to u_{21} (x,t) for the values of the parameters m = 1, μ = 1.5, A = 0, λ = 1 and -3 ≤ x, t ≤ 3 is shown in Fig. 5. Again, for the values m = -0.05, μ = 0.5, A = 4.5, λ = 2, -3 ≤ x, t ≤ 3 and m = 0.05, μ = 0.5, A = 1.5, λ = 1, -3 ≤ x, t ≤ 3, solutions u_{33} (x,t) and u_{35} (x,t) are also given the exact solitary wave solutions of periodic shape. The periodic wave solution is shown in Fig. 7 and 8 respectively.

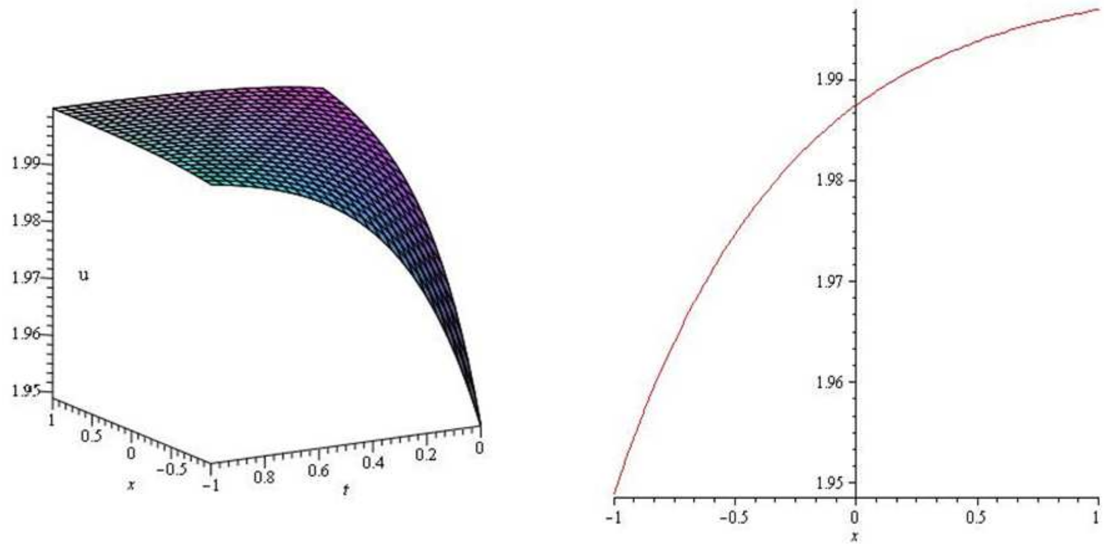

Fig. 1. Kink type soliton profile of Phi-4 equation for m = 0.2, μ = -0.5, A = 2, λ = 1 within -1≤x≤1. (Only shows the shape of $u_1(x, t)$, the left figure shows the 3D plot and the right figure shows the 2D plot for t = 0

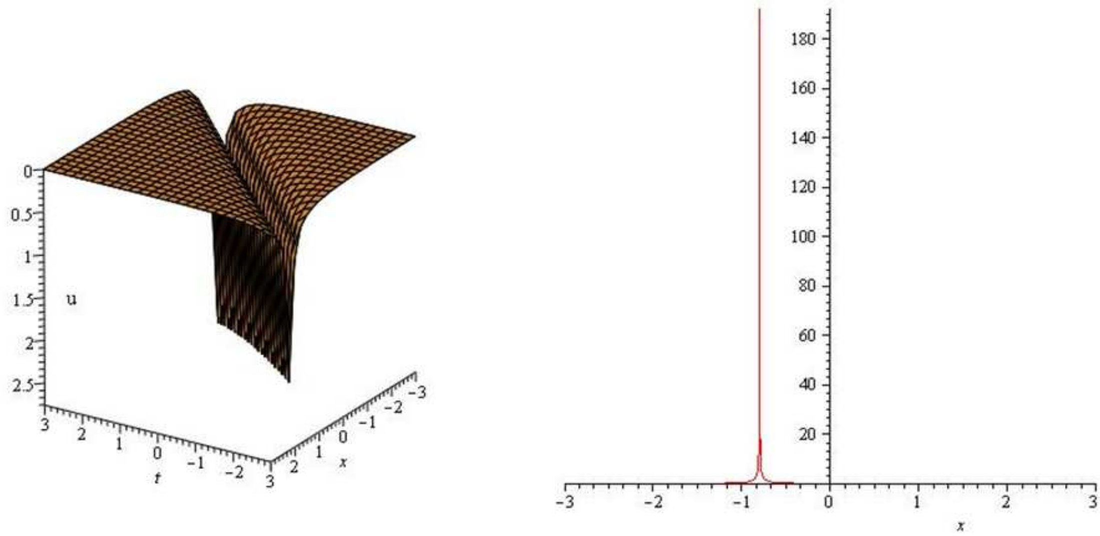

Fig. 2. Singular kink type soliton profile of Phi-4 equation for m =- 0.10, μ =-1.57, A = 1, λ = 1 with -3≤x, t≤3. (Only shows the shape of u_9 (x, t), the left figure shows the 3D plot and the right figure shows the 2D plot for t = 0

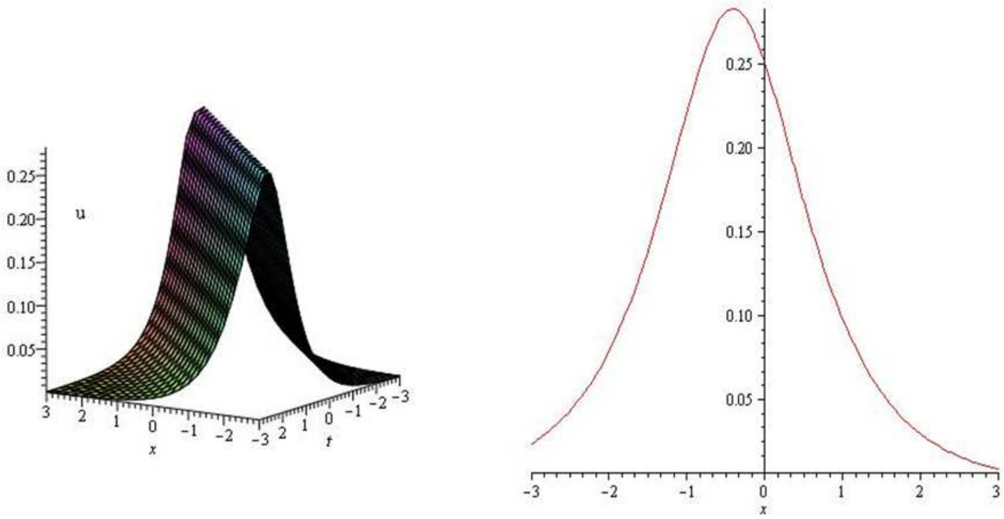

Fig. 3. Bell type soliton profile of Phi-4 equation for $m = 0.2$, $\mu = -1.5$, $A = 0.5$, $\lambda = 1$ with $-3 \le x, t \le 3$. (Only shows the shape of $u_{11}(x, t)$), the left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$

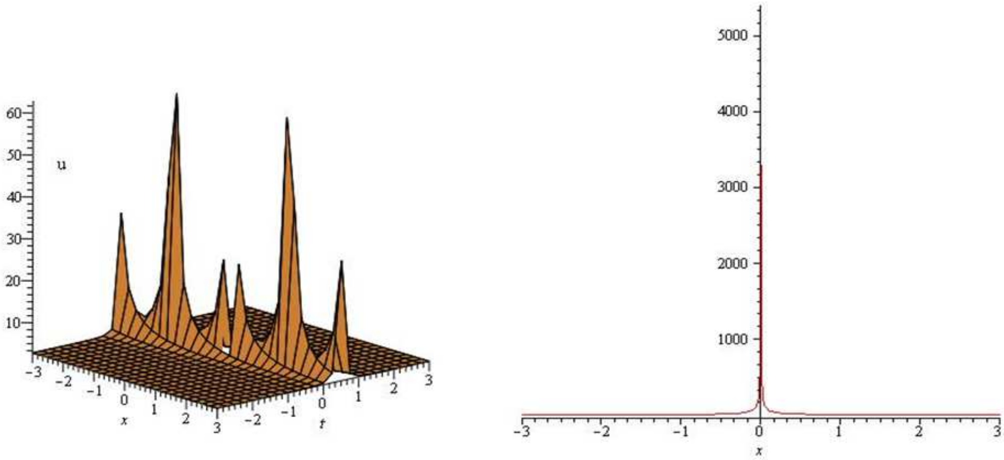

Fig. 4. Singular soliton profile of Phi-4 equation for $m = 4$, $\mu = -0.8$, $A = 0$, $\lambda = 2$ with $-3 \le x, t \le 3$. (Only shows the shape of $u_{17}(x, t)$), the left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$

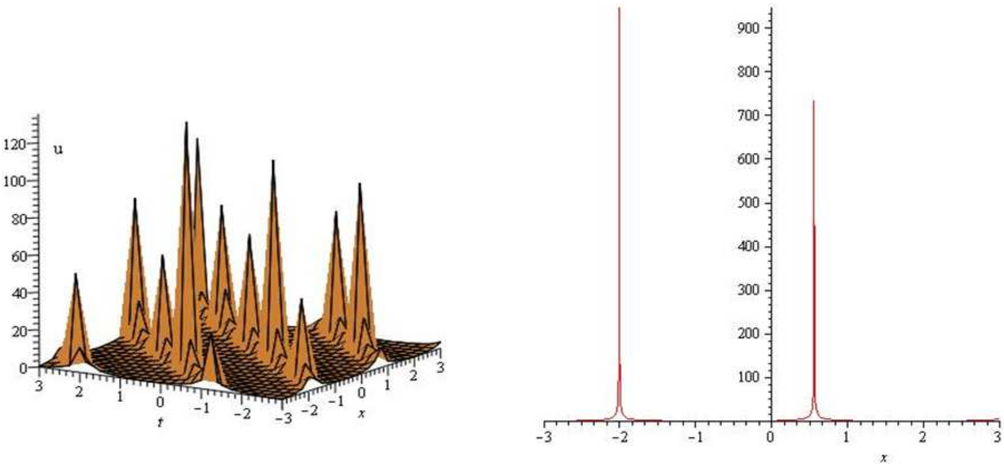

Fig. 5. Periodic wave profile of Phi-4 equation for $m = 1$, $\mu = 1.5$, $A = 0$, $\lambda = 1$ with $-3 \le x, t \le 3$. (Only shows the shape of $u_{21}(x, t)$), the left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$

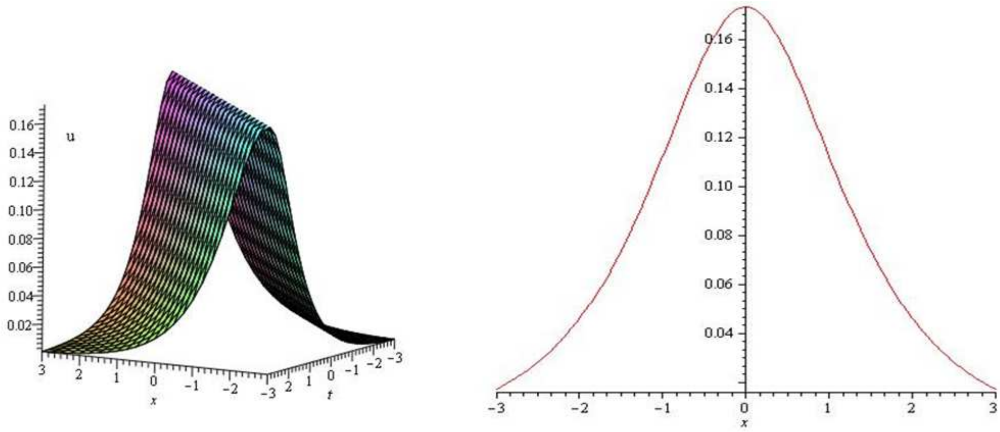

Fig. 6. Bell type soliton profile of Phi-4 equation for m = -0.15, μ = -1, A = 0, λ = 1.5 with -3≤x, t≤3. (Only shows the shape of u_2 (x, t), the left figure shows the 3D plot and the right figure shows the 2D plot for t = 0

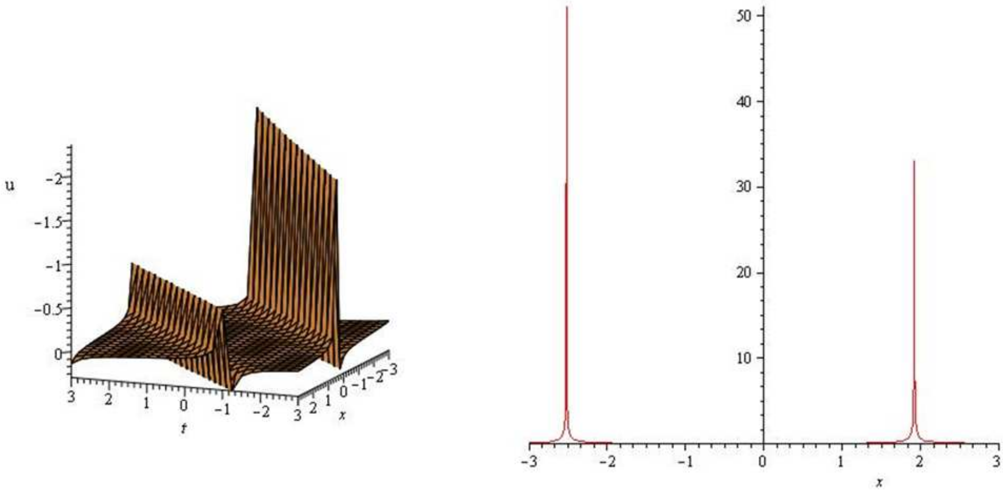

Fig. 7. Periodic wave profile of Phi-4 equation for m = 0.05, μ = 0.5, λ = 2, A = 4.5 with -3≤x, t≤3. (Only shows the shape of u_{33} (x, t), the left figure shows the 3D plot and the right figure shows the 2D plot for t = 0

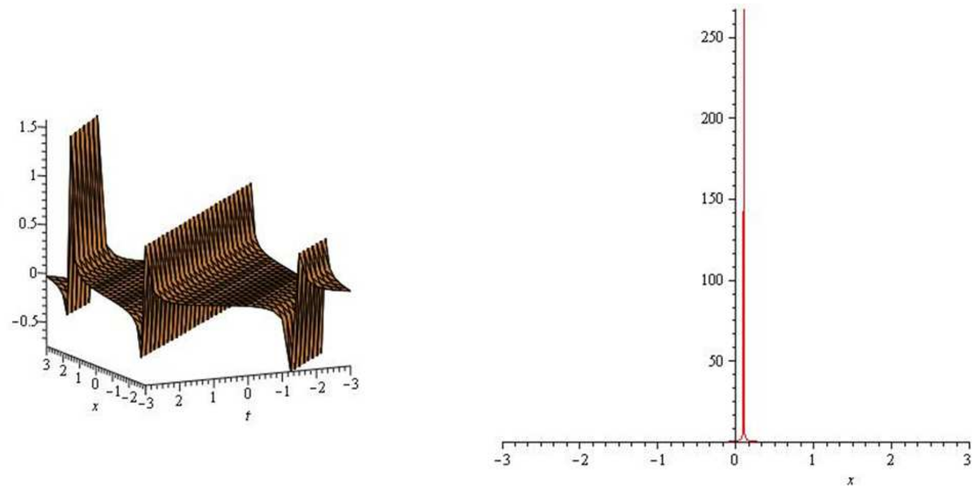

Fig. 8. Periodic wave profile of Phi-4 equation for m = 0.05, μ = 0.5, A = 1.5, λ = 1 with -3≤x, t≤3. (Only shows the shape of u_{33} (x, t), the left figure shows the 3D plot and the right figure shows the 2D plot for t = 0

Graphical Representation

In this sub section, we will plot the figure of the Phi-4 equation by using mathematical software Maple 13. Two and three dimensional plots of the some obtained solutions are shown in Fig. 1-8 to visualize the underlying features of the exact traveling wave solutions of the Phi-4 equation.

Conclusion

In this section, we have seen that two types of traveling wave solutions in terms of hyperbolic and trigonometric functions for the Phi-4 equation is successfully found out by using enhanced (G'/G)-expansion method. From our results obtained in this study, we conclude the enhanced (G'/G)-expansion method is powerful, effective and convenient. The performance of this method is reliable, simple and gives many new solutions. The enhanced (G'/G)-expansion method has more advantages: It is direct and concise. Also, the solutions of the proposed nonlinear evolution equations in this study have many potential applications in nuclear and particle physics. Finally, this method provides a powerful mathematical tool to obtain more general exact solutions of a great many nonlinear PDEs in mathematical physics.

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Ethics

The article is original and contains unpublished material. The authors declare no conflict of interests.

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