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New Analysis for The FGM Thick Cylinders Under Combined Pressure and Temperature Loading

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Abstract: An analytical solution for computing the radial and circumferential stresses in a FGM thick cylindrical vessel under the influence of internal pressure and temperature is presented in this paper. It has been assumed that the modulus of elasticity and thermal coefficient of expansion were varying through thickness of the FGM material according to a power law relationship. Nevertheless the value of the Poisson ratio was taken as constant throughout the material. In the analysis presented here the effect of non-homogeneity in FGM thick cylinder was implemented by choosing a dimensionless parameter, named β, which could be assigned an arbitrary value affecting the stresses in the cylinder. Using Maple 9.5, distribution of stresses in radial and circumferential directions for FGM cylinders under the influence of internal pressure and temperature gradient were obtained. Graphs of variations of stress versus radius of the cylinder were plotted for different values of β. Cases of pressure, temperature and combined loadings were considered separately. It was concluded that by changing the value of β, the properties of FGM could be so modified that the lowest stress levels were reached. The stresses which were produced in FGM and homogeneous material with the same boundary conditions were compared to obtain the optimum value of β.

Key word: FGM, analytical, cylinder, dimensionless

INTRODUCTION

Functionally Graded Materials (FGM) are composite materials with varying properties through thickness. They have thermo-mechanical properties which vary through their thickness and were first conceived by a group of researchers in Japan[1, 2]. The main advantage of such materials which are unique to themselves is the possibility of tailoring the desired properties. Obviously FGM's could be used in a variety of applications which have made them very attractive.

FGMs are fabricated by continuously changing the volume fraction of two basic materials, usually ceramic and metal, in one or more directions. The FGMs that are thus formed exhibit isotropic yet non-homogenous thermal and mechanical properties. These kinds of materials are treated as non-homogenous with material contents that vary continuously along one spatial direction.

Accounts of thermo-elastic and thermo-inelastic problems are given in an extensive review by Noda[3]. Shen[4] has studied the thermal post buckling of functionally graded plates and shells. Solutions to the problem of the uniform heating of a circular cylinder by the Frobenius series method was presented by Zimmerman and Lutz[5]. Obata and Nodo[6] applied the perturbation approach to investigate the thermal stresses in a FGM hollow sphere and in hollow circular cylinder. Ootao and Tanigawa[7] conducted an optimization of the material composition of FGM hollow circular cylinders under thermal loading. A thermo-mechanical analysis, including the coupling effect, for FGM plates and cylinders was presented by Ready and Chin[8]. Tanaka *et al*,[9] gave an improved solution to thermo-elastic materials designed in functionally gradient materials in order to reduce the thermal stresses. They designed FGM property profiles using a sensitivity and optimization method.

Wetherhold *et al*.[10] presented their work for the estimation and control of deformation of FGMs under thermal loadings. Distribution of thermal stress and deformation in functionally graded material shells of revolution under thermal loading due to fluid were investigated by Takezono *et al*.[11]. Zhang *et al*.[12] presented an analytical solution for thermal stresses of axial symmetry functionally gradient materials under steady temperature field. Fukui, Y. and Yamanaka, N[13] considered the analysis of FGM thick-walled

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tubes. Salazar[14] also used finite element method to obtain solutions for functionally graded metal matrix composite tubes.

The problems in solid mechanics involving inhomogeneous media are relatively straight forward. Such problems can be formulated in terms of partial differential equations with variable coefficients by using the basic conservations laws. There has always been difficulty in developing general methods for solving specific boundary value problems. In fact, for the most general second-order partial differential equations with variable coefficients, such general methods do not exist. Because of this difficulty, all existing treatments dealing with the mechanics of inhomogeneous solids are based on a simple function representing material inhomogeneity. For example, in the half-plane elasticity problems considered in[15] and[16], it is assumed that the shear modulus is a power function of the depth coordinate of the form $\mu(y) = \mu_0 y^m$ and the passion's ratio v is constant. It is assumed that the material is isotropic with constant passion's ratio and radially varying elastic modulus is approximated by $E(r) = E_0 r\beta$. Since r is away from zero and ranges in (a, R), by adjusting the constants E_0 and β, it is possible to obtain physically meaningful results. The range $-2 \le 2\beta \le 2$ to be used in the present study covers all the values of coordinate exponent encountered in the references cited earlier[19]. However, these values for β do not necessarily represent a certain material. Various β values are used to demonstrate the effect of inhomogeneity on the stress distribution.

Liew *et al.*[17] presented an analysis of the thermo-mechanical behavior of hollow circular cylinder of functionally graded material. They introduced a novel limiting process that employed the solutions of homogenous hollow circular cylinders. Cylindrical vessels are often used as basic structural components in engineering applications.

Much research has been conducted on isotropic or laminated composite plates and shells[10],[11]. However it seems that very little has been done on FGM thick vessels. Analytical solutions have been given by Johnson and Mellor[18] for thick cylindrical vessels under pressure and temperature loading. Exact solutions for stresses in functionally graded pressure vessels were presented by Tutuncu and Ozturk[19]. They used a material stiffness obeying a simple power law, and determined the inhomogeneity constant which included continuously varying volume fraction of the constituents. Eslami *et al.*[20] gave accounts of their work on the mechanical and thermal stresses in a FGM hollow cylinder due to radially symmetric loads. They assumed the temperature distribution being a function of radius and used a direct method to solve the heat conduction and Navier equations. In a similar work, Eslami, Babaei and Poultangari[21] investigated the thermal and mechanical stresses in a FGM sphere using the same method as in[20]. Hence this paper will develop an analytical model to give solutions for FGM hollow cylindrical vessels that are subject to the action of an arbitrary steady state temperature field, considering pressure effects. This analysis uses the basic equation suggested by references[18], [19] and extends them to include the effect of temperature as well.

THEORY

The stress distribution in a thick-walled cylinder under the influence of internal pressure and temperature loading for homogenous materials, have been formulated by reference[18]. Here the same kind of procedure has been used except that the material is non-homogenous and therefore the properties change as one moves along the radius of the cylinder.

Consider the cross section of a cylindrical pressure vessel as shown in Fig. 1, with the internal radius "a" and external radius "b" and "r" which is normalized as $r = \frac{R}{b}$ where "R" having a value between a and b. Then in order to account for the changing material properties along the radius, a power law relationship[19] is used as follows:

$$\begin{cases} E(r) = E_0 r^{\beta} \\ \alpha(r) = \alpha_0 r^{\beta} \end{cases} \tag{1}$$

Substituting r = 1 in above equation could draw that E_0, α_0 are the modulus of elasticity and thermal coefficient of expansion of outer surface.

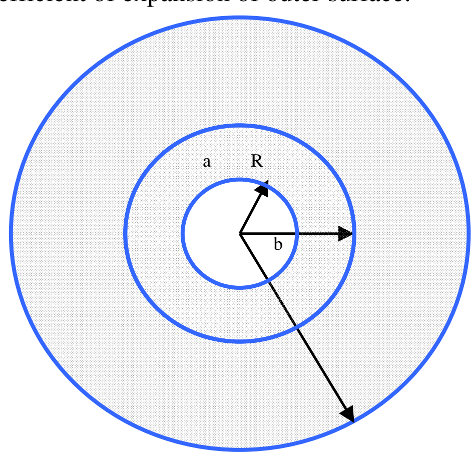

Fig. 1: Cross section of a cylindrical pressure vessel with internal radius "a" and external radius "b"

Note that the value of Poisson's ratio has been taken as constant because although the material is not homogenous but considering that all materials almost have a constant Poisson's ratio value in the elastic range, this is a reasonable assumption.

Now the formulation begins with the expressions for the strain and stress distributions through the thickness of the cylinder using the equilibrium equation as follows:

$$\frac{d\sigma_r}{dr}+\frac{(\sigma_r-\sigma_\theta)}{r}=0 \qquad (2)$$

As this is a plane strain problem and hence the longitudinal strain is zero, the radial and circumferential strains are given in terms of displacements, as:

$$\begin{cases} e_r = u' = \dfrac{du}{dr} \\ e_\theta = \dfrac{u}{r} \end{cases} \qquad (3)$$

It could easily be shown that the stresses in the radial and hoop directions are given by:

$$\begin{cases} \sigma_\theta = \dfrac{E}{(1+\nu)(1-2\nu)}\left[\nu e_r + e_\theta(1-\nu)-(1+\nu)\alpha T\right] \\ \sigma_r = \dfrac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)e_r+\nu e_\theta-(1+\nu)\alpha T\right] \end{cases} \qquad (4)$$

However if in Eq. (2) we substitute for stresses from Eq. (4), and also substitute for strain from (3), the following equation could be obtained for displacement:

$$r^2u''+r(B+1)u'+(\nu s\beta-1)=C_2\frac{f'(r)}{r^{\beta-2}} \qquad (5)$$

Where

$$C_2=\frac{\alpha_0(1+\nu)}{(1-\nu)},\nu s=\frac{\nu}{(1-\nu)},f(r)=r^{2\beta}T$$

This is a non homogenous form of Euler-Cauchy equation. By using the variation of parameters method (Lagrange) we have the following solution:

Considering the solutions of homogeneous form of this equation which was presented in[19], we know that general solution of homogenous form of above Euler-Cauchy equation is:

$$\begin{cases} y_1 = r^{s_1} \\ y_2 = r^{s_2} \end{cases} \Rightarrow y_h = y_1 + y_2$$

Where S_1 and S_2 are given by:

$$\begin{cases} s_1 = \dfrac{1}{2}(-\beta-\sqrt{4+\beta^2-4\beta\nu s}) \\ s_2 = \dfrac{1}{2}(-\beta+\sqrt{4+\beta^2-4\beta\nu s}) \end{cases}$$

Variation of parameters method was used for achieving the particular solution which is presented as flows:

$$y_p=-C_2\int\frac{f'(r)r^{s_2}}{r^\beta r^{-\beta-1}(s^*)}dr+C_2\int\frac{f'(r)r^{s_1}}{r^\beta r^{-\beta-1}(s^*)}dr \qquad (6)$$

Where

$$s^*=s_2-s_1=\sqrt{4+\beta^2-4\beta\nu s}$$

Also the temperature gradient is given by:

$$T=\frac{t_a}{\ln(\frac{1}{w})}\ln(\frac{1}{r}) \qquad (7)$$

Note that above relation is normalized form of temperature gradient which was presented in[18] and "w" equals to $\frac{a}{b}$.

Substituting (7) into (6) and using Eq. (5) after simplifying, particular solution was obtained:

$$y_p=\alpha_0 t_a r^{B+1}(m_1\ln(\frac{1}{r})-m_2) \qquad (8)$$

Where m_1 and m_2 are given by:

$$m_1=(\frac{1+\nu}{1-\nu})\frac{2\beta}{(2\beta+s_2+1)(2\beta+s_1+1)\ln(\frac{1}{w})}$$

$$m_2=(\frac{1+\nu}{1-\nu})\frac{(s_2s_1+s_2+s_1+1-4\beta^2)}{(2\beta+s_2+1)^2(2\beta+s_1+1)^2\ln(\frac{1}{w})}$$

Solution for the problem in fact should be sum of general and particular solutions and could be given for the displacements in terms of which strains and stresses are defined:

$$Y=y_h+y_p\Rightarrow u=Gr^{s_1}+Hr^{s_2}+\alpha_0 t_a r^{B+1}(m_1\ln(\frac{1}{r})-m_2) \qquad (9)$$

In above equation G and H are unknown constants which were obtained by applying boundary condition and following the below process:

The boundary conditions are also given by:

$$\begin{cases} r = w = a/b & \Rightarrow \sigma_r = -p \\ r = 1 & \Rightarrow \sigma_r = -0 \end{cases}$$

Now using the boundary conditions and substituting for u from (9) into (3) and subsequently (3) into (4), we arrive at a set of equations with some unknown constants which were called G and H and after solving these equations they are given as follows:

$$G = -\frac{n_1 w^{s_2} + n_2 - \frac{p(1+\nu)(1-2\nu)}{E_0}}{(-s_1 + \nu s_1 - 2\nu)(-w^{s_2} + w^{s_1})}$$
$$H = \frac{n_1 w^{s_1} + n_2 - \frac{p(1+\nu)(1-2\nu)}{E_0}}{(-s_1 + \nu s_1 - 2\nu)(-w^{s_2} + w^{s_1})} \qquad (10)$$

Where again n_1 and n_2 are given by:

$$n_1 = -t_a m_1 \alpha_0 (-m_3\beta - m_3 + m_3\beta\nu - 2\nu - \beta - m_3\nu + \beta\nu)$$
$$n_2 = m_1 w^{\beta} t_a \alpha_0 (-2\nu + \beta\nu - \beta) + w^{(\beta+1)} t_a \alpha_0 (-m_1 m_3 \beta - m_1 m_3 + m_1 m_3 \beta\nu - m_1 m_3 \nu + 1 + \nu)$$

Therefore substituting G, H back in the displacement Eq. (9) and substituting the obtained equation of displacement back in strain and subsequently stress equations, the final relationship for the distribution of stresses could be obtained as follows:

$$\sigma_r = -\frac{r^{\beta}}{r(1+\nu)(1-2\nu)(1-w)}\Big((1-w)E_0\alpha_0 t_a m_1 r^{\beta} (-m_3 r + \nu\beta - 2\nu - \beta - m_3 r\beta + m_3 r\nu\beta - m_3 r\nu) Gr^{s_1} E_0 (1-w)(-s_1 + \nu s_1 - 2\nu) + Hr^{s_2} E_0 (1-w)(-s_2 + \nu s_2 - 2\nu) + \alpha_0 w t_a (1-r)(1-2\nu)(1+\nu)^2\Big)$$

$$\sigma_\theta = -\frac{E_0 r^{\beta}}{r(1+\nu)(1-2\nu)(1-w)} \Big(Gr^{s_1}(w-1)(\nu s_1 + 1) + Hr^{s_2}(w-1)(\nu s_2 + 1) + w(1-r)(1+\nu) + \alpha_0 t_a r^{\beta}\Big(m_1(w-1)\big((1+\nu\beta)(rm_3 + 1) + r\nu m_3\big) \qquad (11)$$

Stresses for the cylindrical FGM pressure which obtained above will be compared with those for the homogenous ones. The stress expressions for homogenous cylinder are presented below. Note that these are normalized form of what were mentioned in[18].

$$\sigma_r^H = -\frac{\alpha_0 E_0 t_a w (r-1)(rw + w + r)(w - r)}{r^3(\nu - 1)(w - 1)(1 + w + w^2)}$$
$$\sigma_\theta^H = \frac{\alpha_0 E_0 t_a w (r^2(w+1)(2r-1) - w^2(r^2+1))}{2r^3(\nu - 1)(w - 1)(1 + w + w^2)} \qquad (12)$$

Above expressions can also easily be obtained by setting $\beta = 0$ in the FGM case (of course considering computational errors, we choose $10^{-10}$for β instead of zero).

RESULTS AND DISCUSSION

The analytical solution presented in the previous section was applied to a thick hollow cylinder of inner radius a = 0.75m and outer radius of b = 1m. The modulus of elasticity and the thermal coefficient of expansion at the outer surface of the cylinder were taken as E_0 = 200Gpa and α_0 = 1.2 $\times 10^{-6}$/°C respectively. Poisson ratio assumed to be constant (ν = 0.3). The internal pressure was taken to be equal to 400Mpa and the external pressure was assumed to be zero. The values of temperature at the inner and outer surfaces were assumed to be t_a = 1000°C and t_b = 0°C respectively.

Pressure only: The results for internal pressure loading were normalized with respect to the homogenous cylinder to study the impact of the inhomogeneity on the results for the FG cylinder. The distribution of radial stress for different values of β is seen in Fig. 2.

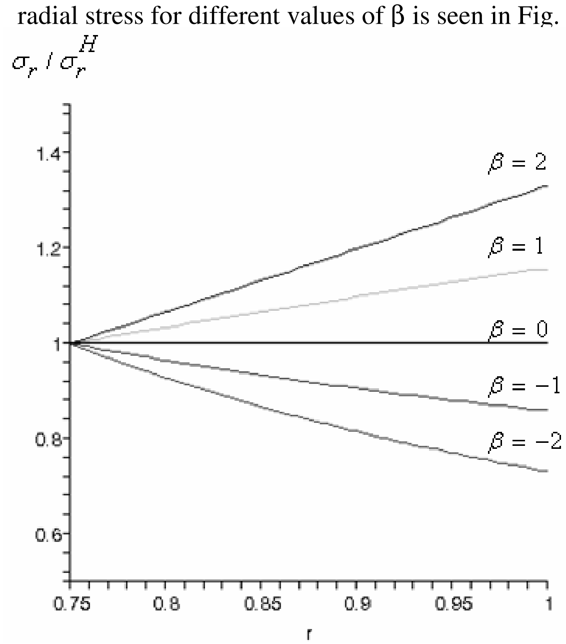

Fig. 2: variations of normalized radial stress in a cylindrical vessel under the loading of pressure

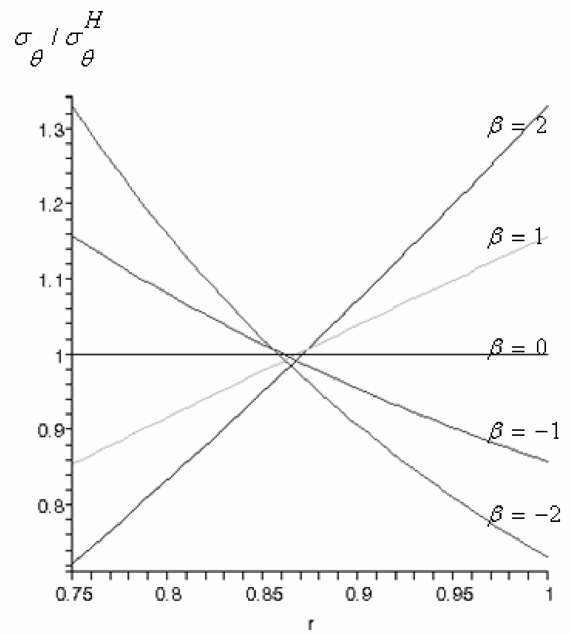

Fig. 3: Distribution of circumferential stress in cylindrical vessel under the loading of pressure

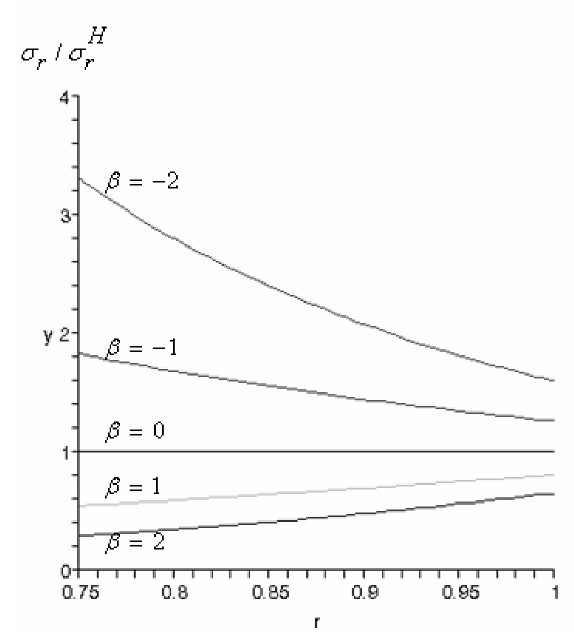

Fig. 4: Variations of normalized radial stress in a cylindrical vessel under the loading of temperature

Here a higher value of β means increasing stiffness (see Eq. (1)). It could be seen from Fig. 2. that for higher positive β values the stress in the radial direction increases while for a negative value of β it decreases. In Fig. 3 for positive values of β the maximum hoop stress occurs at the outer surface of the cylinder while for negative values of β the maximum happens at the inner surface. At a radial distance of r = 0.87 the stress values for all values of β converge towards the stress values in the homogenous material (β = 0).

Temperature only: The effect of thermal loading as seen in Fig. 4 and 5, indicates the fact that just like the behaviour of homogenous cylinder, in FG cylinder too, the stress distribution trend for the thermal loading is opposite to that due to internal pressure. However for positive values of β the distribution of radial stress as seen in Fig. 4. indicates a fairly uniform trend as compared to that obtained for the negative values of β. Also for negative values of β much higher values are observed at the inner surface of the cylinder. As shown in Fig. 5. the circumferential stress in general increases and becomes a maximum at the outer surface of the cylinder. For larger values of β lower stress values are observed. It could also be seen form the same figure that there is a sudden drop at about the mid-surface (r = 0.85 to 0.9) of the cylinder wall depending on the β values.

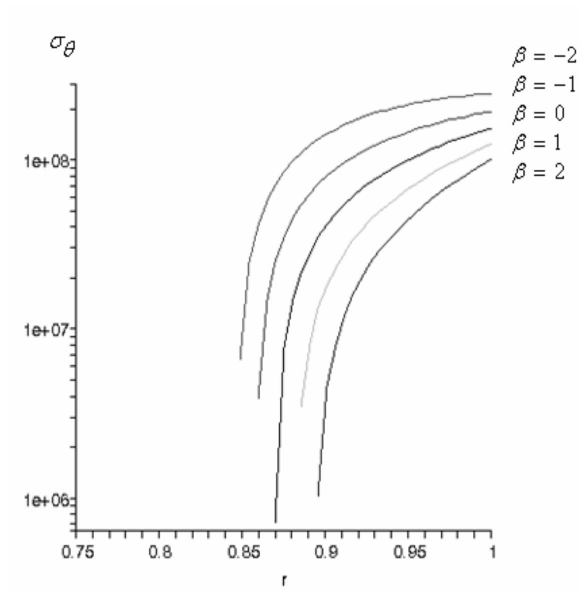

Fig. 5: variations of circumferential stress in a cylindrical vessel under the loading of temperature

Combined loading (pressure + temperature): Since both pressure and temperature loadings are in the elastic range, the principle of superposition was applied for the combined loading and the following results were obtained.

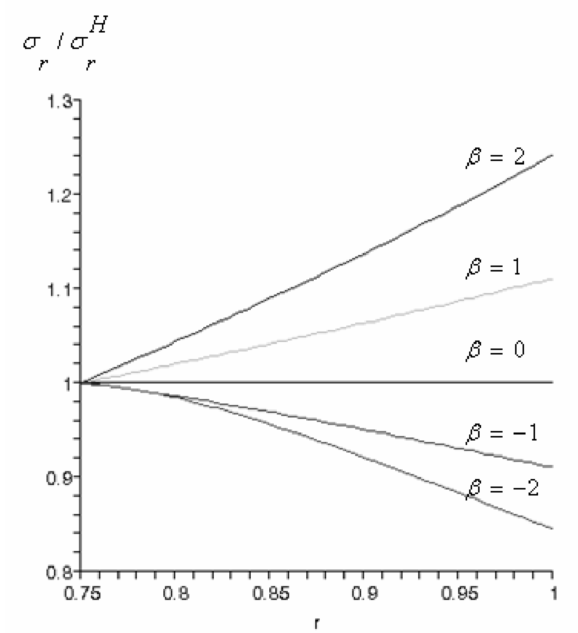

Fig. 6: variations of normalized radial stress in a cylindrical vessel under the combined loading of pressure and temperature

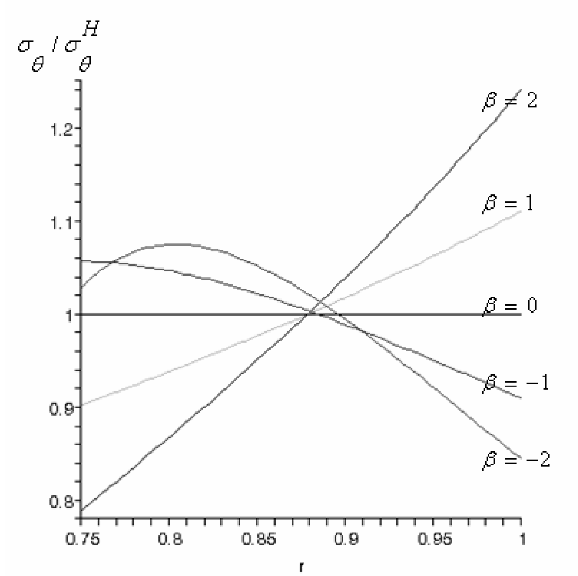

Fig. 7: variations of normalized circumferential stress in a cylindrical vessel under the combined loading of pressure and temperature

In Figure 6, normalised values of radial stresses were plotted against radial positions in the cylinder walls. However for higher values of β larger stress values are observed at the outer surface of the cylinder while for the lower values of β smaller stress values are

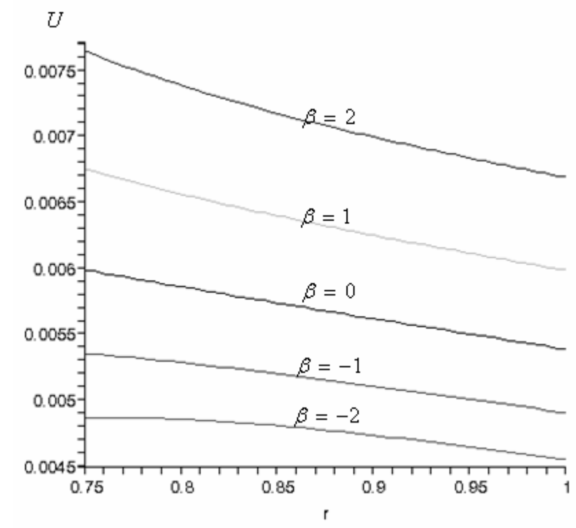

Fig. 8: Displacement in a cylindrical vessel under the combined loading of pressure and temperature

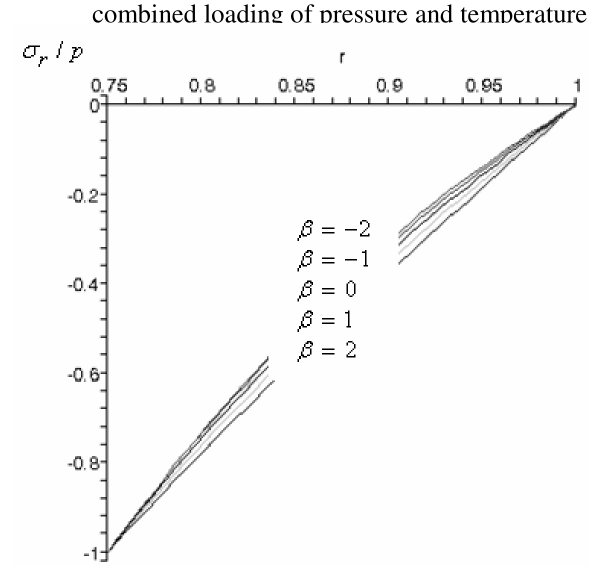

Fig. 9: Distribution of relative radial stress in a cylindrical vessel under the combined loading of pressure and temperature

obtained. This could be interpreted physically meaning that the outer surface of the cylinder is biased to bear the stress due to its higher stiffness.

In Figure 7, it could be noted that the combined effect of temperature and pressure loading on the hoop stress also produces the same stress values at about r = 0.88. Here again higher β values give higher stresses at the outer surface of the cylinder while they produce lower stresses at the inner surface. The radial displacement could be observed in Fig. 8 to have reduced at the outer surface of the cylinder with respect to the inner surface but for higher β values larger

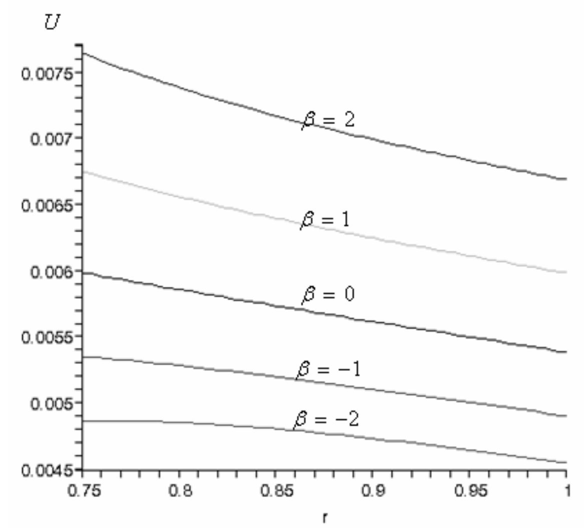

Fig. 10: Distribution of relative hoop stress in a cylindrical vessel under the combined loading of pressure and temperature

displacements are possible. Figure (9) shows the normalised values of radial stress with regard to the internal pressure increase from inner to outer surface and not showing much difference for different values of β. Similar results for the relative hoop stress with respect to internal pressure are also shown in Fig. 10. Higher positive β values give higher stresses at the outer surface and negative values give lower stresses at the same position.

From above results it could be said that the FGM exponent of Eq. (1) could be used as a very useful design parameter for tailoring the stress distributions to fit to the specific applications.

CONCLUSION

The following conclusions could be drawn from the work presented in this paper:

1) A closed form exact analytical solution was obtained for the combined pressure and temperature loading of FGM circular cylinder.
2) Depending on applied boundary condition, by selecting optimum value of β, desirable level of radial and circumferential stresses could be obtained in FGM cylinders with respect to those in homogenous ones.
3) By setting β = 0 in Eq. (11), radial and circumferential stresses expressions turned to homogenous ones which could assent the validity of formulation.
4) For further work the present formulation could be extended and developed for the yield region so that an elastic plastic solution could be obtained.
5) Another possible line for further research is the assumed power law for the FGM properties which could perhaps be replaced by a more realistic form with regard to the manufacturing problems of FGM cylinders.

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