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Solving Volterra's Population Model Using New Second Derivative Multistep Methods

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Abstract: In this study new second derivative multistep methods (denoted SDMM) are used to solve Volterra's model for population growth of a species within a closed system. This model is a nonlinear integro-differential where the integral term represents the effect of toxin. This model is first converted to a nonlinear ordinary differential equation and then the new SDMM, which has good stability and accuracy properties, are applied to solve this equation. We compare this method with the others and show that new SDMM gives excellent results.

Key words: multistep and multi-derivative methods, volterra's population model, integro-differential equation, stiff systems of ODEs

INTRODUCTION

Volterra's model for the population growth of a species within a closed system is given in[8,9,10] as

$$\frac{dp}{d\tilde{t}} = ap - bp^2 - cp\int_0^{\tilde{t}} p(x)dx, \quad p(0) = p_0, \tag{1}$$

Where $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient and $c > 0$ is the toxicity coefficient. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long term. p_0 is the initial population and $p = p(\tilde{t})$ denotes the population at time $\tilde{t}$.

This model is a first-order integro-ordinary differential equation where the term $cp\int_0^{\tilde{t}} p(x)dx$ represents the effect of toxin accumulation on the species. We apply scale time and population by introducing the nondimensional variables

$$t = \frac{\tilde{t}c}{b}, u = \frac{pb}{a}$$

to obtain the nondimensional problem:

$$\kappa\frac{du}{dt} = u - u^2 - u\int_0^t u(x)dx, \quad u(0) = u_0 \tag{2}$$

Where $u(t)$ is the scaled population of identical individuals at time t and $\kappa = c/(ab)$ is a prescribed non-dimensional parameter. The only equilibrium solution of eq.2 is the trivial solution $u(t) = 0$ and the following analytical solution[10] shows that $u(t) > 0$ for all t if $u_0 > 0$.

$$u(t) = u_0\exp(\frac{1}{\kappa}\int_0^t [1 - u(\tau) - \int_0^\tau u(x)dx]d\tau)$$

In[9], the singular perturbation method for Volterra's population model is considered. This author scaled out the parameters of eq.1 as much as possible and considered four different ways to do this. He considered two cases $\kappa = c/(ab)$ small and $\kappa = c/(ab)$ large.

It is shown in[9] that for the case $\kappa \ll 1$, where populations are weakly sensitive to toxins, a rapid rise occurs along the logistic curve that will reach a peak and then is followed by a slow exponential decay. And, for large κ, where populations are strongly sensitive to toxins, the solutions are proportional to $\text{sech}^2(t)$.

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In[5] Adomian decomposition method and Sinc-Galerkin method compared for the solution of some mathematical population growth models. This showed that Adomian decomposition method is more efficient and easy to use for the solution of Volterra's population model. In[11], the series solution method and the decomposition method are implemented independently to eq.2 and to a related nonlinear ordinary differential Equation. Furthermore, the Pade approximations are used in the analysis to capture the essential behavior of the populations $u(t)$ of identical individuals, also the approximation of u_{max} and exact value of u_{max} for different κ were compared.

The solution of eq.1 has been of considerable concern. Although a closed form solution has been achieved in[8,9], it was formally shown that the closed form solution cannot lead to any insight into the behavior of the population evolution[8]. In the literature several numerical solutions for Volterra's population model have been reported. In[8], the successive approximations method was suggested for the solution of eq.2, but was not implemented. In this case the solution $u(t)$ has a smaller amplitude compared to the amplitude of $u(t)$ for the case $\kappa \ll 1$. In[10], several numerical algorithms namely the Euler method, the modified Euler method, the classical fourth-order Runge-Kutta method and Runge-Kutta-Fehlberg method for the solution of eq.2 are obtained. Moreover, a phase-plane analysis is implemented. In[10], the numerical results are correlated to give insight on the problem and its solution without using perturbation techniques. However, the performance of the traditional numerical techniques is well known in that it provides grid points only ,and in addition, it requires a large amounts of calculations. The authors of[6,7] applied spectral method to solve Volterra's population on a semi-infinite interval. This approach is based on a Rational Tau method. They obtained the operational matrices of derivative and product of rational Chebyshev and Legendre functions and then they applied these matrices together with the Tau method to reduce the solution of this problem to the solution of system of algebraic Equations.

On the other hand, in recent years, numerous works have been focusing on the development of more advanced and efficient methods for initial value problems especially for stiff systems. For example, as Enright[2] used second derivative of solution in his algorithm, Cash[1] and Ismail[4] introduced second derivative multistep methods that have good stability properties. These methods are A-stable of high orders. One of these efficient methods that have good stability and accuracy properties is a new class of second derivative multistep methods that is introduced by Hojjati *et al.*[3]. The main superiority of this new class of methods lead us to apply this new class of methods to solve Volterra's population model after converting it to a system of ODEs.

This study is arranged as follows: in the first section we describe new second derivative multistep methods. In the second section Volterra's population model is considered. This equation is first converted to an equivalent nonlinear ordinary differential equation and then our method can be applied to solve this new equation. In the next section the proposed method is applied to several numerical examples and a comparison is made with existing methods that were reported in the literature to solve similar problems. The numerical results and advantages of the method are discussed in the final section.

MATERIALS AND METHODS

New second derivative multi step methods: Let us consider the stiff initial value problem

$$y'(t) = f(t, y(t)), y(t_0) = y_0 \qquad (3)$$

on the finite interval

$$I = [t_0, t_N] \text{ where } y : [t_0, t_N] \to IR^m$$

and

$f : [t_0, t_N] \times IR^m \to IR^m$ is continuous.

One of the main directions of search for higher order A-stable multi step methods is the use of higher derivatives of the solutions. By applying the second derivative of solution in algorithm of multistep methods, a new class of methods are introduced. These methods are known as second derivative multi step methods (SDMM).

The general SDMM can be written in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j g_{n+j} \qquad (4)$$

Where $\alpha_j, \beta_j, \gamma_j$ are parameters to be determined and $g_{n+j} = f^{(1)}_{n+j}$. Taylor expansion shows that the method equation 4 is of order p if and only

$$\sum_{j=0}^{k} \alpha_j j^q = q\sum_{j=0}^{k} \beta_j j^{q-1} + q(q+1)\sum_{j=0}^{k} \gamma_j j^{q-2} \qquad (5)$$

with $0 \leq q \leq p$. Some known important SDMM schemes were introduced by Enright[2], Cash[1], and Ismail[4] that are A-stable methods with high order of accuracy. New second derivative multistep methods that were introduced by Hojjati *et al*.[3] are part of a new class of similar methods that have good accuracy, good stability properties and are suitable for solving stiff equations.

The new SDMM takes the following general form

$$\sum_{j=0}^{k}\hat{\alpha}_j y_{n+j} = h\hat{\beta}_k f_{n+k} + h^2(\hat{\gamma}_k g_{n+k} - \hat{\gamma}_{k+1} g_{n+k+1}), \tag{6}$$

Where

$$g(x,y) = y'' = f_x + f_y f$$

and

$$\hat{\alpha}_k = 1$$

and the other coefficients are chosen so that equation 6 has order $k+2$. The coefficients of k-step methods of class equation 6 are given in[3].

Assuming that the solution values $y_n, y_{n+1}, \ldots, y_{n+k-1}$ are available, the way in which equation 6 is used in practice is as follows:

Stage 1: Compute $\bar{y}_{n+k}$ as the solution of

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\beta_k f_{n+k} + h^2\gamma_k g_{n+k} \tag{7}$$

Where $\alpha_k = 1$ and the other coefficients are chosen so that equation 7 has order $k+1$. The coefficients of k-step methods of class equation 7 are given in[3].

Stage 2: Compute $\bar{y}_{n+k+1}$ as the solution of

$$\sum_{j=0}^{k}\alpha_j y_{n+j+1} = h\beta_k f_{n+k+1} + h^2\gamma_k g_{n+k+1}, \tag{8}$$

Stage 3: Evaluate

$$\bar{g}_{n+k+1} = g(x_{n+k+1}, \bar{y}_{n+k+1})$$

Stage 4: Compute y_{n+k} as the solution of

$$\sum_{j=0}^{k}\hat{\alpha}_j y_{n+j} = h\hat{\beta}_k f_{n+k} + h^2(\hat{\gamma}_k g_{n+k} - \hat{\gamma}_{k+1}\bar{g}_{n+k+1})$$

The method is a k-step method of order $k+2$. It is A-stable up to order 9. For more details see[3].

Solving Volterra's population model: In this section we study the algorithm of solving Volterra's population model by using the new SDMM. We first convert Volterra's population model equation 2 to an equivalent nonlinear ordinary differential equation. Let

$$y(t) = \int_0^t u(x)dx \tag{9}$$

This leads to

$$y'(t) = u(t), y''(t) = u'(t) \tag{10}$$

Inserting eq.9-eq.10 into 2 yields the nonlinear differential equation as following.

$$\kappa y''(t) = y'(t) - (y'(t))^2 - y(t)y'(t) \tag{11}$$

with the initial conditions

$$y(0) = 0 \tag{12}$$

$$y'(0) = u_0 \tag{13}$$

that obtained by using eq.9 and eq.10, respectively. The second order differential equation with initial values eq.12 and eq.13 now can be considered as following first order initial value problem

$$\begin{aligned} z_1' &= z_2, & z_1(0) = 0, \\ z_2' &= (\frac{1}{\kappa})(z_2 - z_2^2 - z_1 z_2), & z_2(0) = u_0, \end{aligned}$$

where

$$z_1(t) = y(t),$$
$$z_2(t) = y'(t)$$

Now we can apply the new second derivative multi step methods to this system. It can be seen that the above system of ODEs will be stiff for small κ.

Table 1: A comparison among methods in[6,7,11] and the present method with the exact values for u_{max}

κ	Method in[6]	Method in[7]	Method in[11]	Present Method	Exact u_{max}
0.02	0.923327	0.923463	0.90383805	0.92342714	0.92342717
0.04	0.873605	0.873708	0.86124018	0.87381998	0.87371998
0.1	0.769623	0.769734	0.76511308	0.7697414	0.76974149
0.2	0.658872	0.659045	0.65791231	0.65905037	0.65905038
0.5	0.485076	0.485188	0.48528235	0.48519029	0.48519030

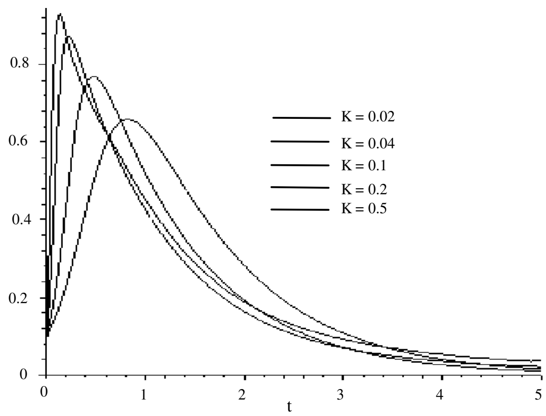

Fig. 1: The results of new SDDM calculation for κ = 0.02, 0.04, 0.1, 0.2, 0.5

So, it is necessary to use a numerical method with extended stability.

RESULTS AND DISCUSSION

Illustrative example: In this study we applied the method presented to examine the mathematical structure of $u(t)$. In particular, we studied the rapid growth along the logistic curve that will reach a peak, then followed by the slow exponential decay where $u(t) \to 0$ as $t \to \infty$. The mathematical behavior introduced by[8] and justified by[9] using singular perturbation methods for the inner and outer solutions. Further, these properties were also confirmed by[10] upon using a phase plane analysis and[11] by using Pade approximations. We applied the method presented in this study and solved 2 for $u_0 = 0.1$ and $\kappa = 0.02, 0.04, 0.1, 0.2$ and 0.5. The numerical solutions obtained using the method described, as well as the method reported in[6,7,11] are compared with the exact solutions in Table 1.

CONCLUSION

To achieve accurate solutions to problems, some methods with extensive regions of stability were applied. We converted Volterra's population model to a system of ODEs. The new SDMM algorithms proposed in this study solve stiff systems effectively. In Table 1 we compare the solutions of these methods for u_{max} and compare the results with exact solutions and also the other models.

The new SDMM provides accurate and numerically stable solutions for different κ s.

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