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On Quasi E-Convex Bilevel Programming Problem

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Abstract: Bilevel programming problems involve two optimization problems where the data of the first one is implicity determined by the solution of the second. This study introduces the notions of E-convexity and quasi E-convexity in bilevel programming problems to generalize quasi convex bilevel programming problems.

Key words: Bilevel Programming, E-convex Sets, Quasi E-convex Functions, Extreme Point, Optimal Solutions

INTRODUCTION

Multilevel programming has been proposed for dealing with hierarchical systems. It is characterized by the existence of two or more optimization problems in which the constraint region of each level problem is implicitly determined by another optimization problem. Due to its complexity, the bilevel case has been considered mainly. It can be formulated as follows:

$$\max_{x_1 \in X_1} f_1(x_1, x_2) \quad (1a)$$

Where, x_2 solves

$$\max_{x_2 \in X_2} f_2(x_1, x_2) \quad (1b)$$

$$\text{subject to } (x_1, x_2) \in S \quad (1c)$$

Where, $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are the variables controlled by the first-level decision maker and the second-level decision maker, respectively $f_1, f_2 : R^n \to R$, $n=n_1+n_2$ and

$$S = \{ x = (x_1, x_2) \in R^n : g_j(x) \le 0, j = 1, ..., m \}.$$

This is a non convex optimization problem that has received increasing attention in the literature. Most results in this field have been obtained assuming that all functions are linear. In this case, it is proved, that the solution to the problem must occur at an extreme point of the region S, which is a polyhedron [1, 2]. Based on this fact, several algorithms have been proposed which find the optimal solution using enumerative schemes [3]. Omar Ben-Ayed [4] provides a survey on the linear bilevel problem. In the nonlinear case, it is usually assumed that the second-level objective function f_2 and g_j functions involved are convex. Bialas [3] and Fortuny-Amat [5] replace the second-level problem with its Kuhn-Tucker conditions as a solution technique for the problem. Calvete and Gale [6] focus on a special case of (1) in which functions f_1 and f_2 are quasi concave and the feasible region, given by the common constraints to both problems, is a polyhedron and prove that there is an extreme point of the feasible region S which solves the problem, thus extending the result obtained for the linear bilevel problem.

The concept of E-convexity were introduced by Youness [7, 8], which has some important applications in various branches of mathematical sciences, is introduced to extend quasiconvex bilevel programming problem to quasi E-convex bilevel programming problem.

In this study we prove some properties of E-convex sets and quasi E-convex functions and discuss the solution of a special case of problem (1) in which functions f_1 and f_2 are quasi E-convex and the feasible region, given by the common constraints to both problems, is a polyhedron. We prove that there is an extreme point of the feasible region S which solves the problem. This result is considered a generalization to the result in Calvete and Gate [6].

Quasi E-Convex Function

Definition: A set $M \subseteq R^n$ is said to be E-convex set with respect to an operator $E : R^n \to R^n$ if and only if $\lambda E(x) + (1-\lambda)E(y) \in M$ for each $x, y \in M$ and $\lambda \in [0, 1]$ [7].

Definition: A function $f : R^n \to R$ is said to be quasi E-convex function, with respect to an operator $E : R^n \to R^n$, on an E-convex set $M \subset R^n$ if and only if $f[\lambda E(x) + (1-\lambda)E(y)] \le \max\{ f \circ E(x), f \circ E(y)\}$ for any $x, y \in M$, $\lambda \in [0, 1]$ [9].

The function f is said to be quasi E-concave if and only if, $f[\lambda E(x) + (1-\lambda)E(y)] \ge \min\{ f \circ E(x), f \circ E(y)\}$ for any $x, y \in M$, $\lambda \in [0, 1]$.

Definition: Let X be a vector space. A linear operator $E : X \to X$ is called a projection in X if $E^2 \equiv E$, i.e., if $E(E(x)) = E(x)$ for every $x \in X$ [10].

Definition: Let X be a vector space, $E : X \to X$ be a linear operator and $f : X \to R$ be a functional. The operator E is said to preserve the order of f in the sense that $f \circ E(x_1) \ge f \circ E(x_2)$ whenever $f(x_1) \ge f(x_2)$, $x_1, x_2 \in X$.

Definition: Let X be a real vector space and $E: X \to X$ be an operator. A vector sum $\lambda_1 E(x_1) + \lambda_2 E(x_2) + ... + \lambda_n E(x_n)$ is called an E-convex combination of $x_1, x_2, ..., x_n \in X$ if the coefficients λ_i are all non-negative and $\sum_{i=1}^{n} \lambda_i = 1$.

It is also called convex combination of $E(x_1), E(x_2), ..., E(x_n)$.

Proposition: Let $E: R^n \to R^n$ be a projection map in R^n and M be a nonempty E-convex subset of R^n. If $f: R^n \to R$ is a quasi E-convex function on M, then an $E\alpha$-level set $K_\alpha^E = \{x \in M : f \circ E(x) \le \alpha, \alpha \in R\}$ is an E-convex set.

Proof: Assume $x, y \in K_\alpha^E$, then we have $x, y \in M$ and $\max\{f \circ E(x), f \circ E(y)\} \le \alpha$. So $\lambda E(x) + (1-\lambda)E(y) \in M$ for all $\lambda \in [0,1]$. Since f is quasi E-convex, then $f[\lambda E(x) + (1-\lambda)E(y)] \le \max\{f \circ E(x), f \circ E(y)\} \le \alpha$. Since E is a projection in R^n, $f[\lambda E^2(x) + (1-\lambda)E^2(y)] \le \alpha$ and so $(f \circ E)[\lambda E(x) + (1-\lambda)E(y)] \le \alpha$. Hence the result.

Lemma: Let $E: R^n \to R^n$ be a projection map in R^n and M be a nonempty E-convex subset of R^n. If $f: R^n \to R$ is a quasi E-convex function on M, then $K_\alpha^E \subset K_\alpha = \{x \in M : f(x) \le \alpha\}$.

Proof: Let $\bar{x} \in K_\alpha^E$. From the E-convexity of K_α^E, there are two points $x, y \in K_\alpha^E$ such that $\bar{x} = \lambda E(x) + (1-\lambda)E(y)$, for some $\lambda \in [0,1]$. Since f is quasi E-convex, then $f(\bar{x}) = f[\lambda E(x) + (1-\lambda)E(y)] \le \max\{f \circ E(x), f \circ E(y)\} \le \alpha$. Therefore, $\bar{x} \in M$ and $f(\bar{x}) \le \alpha$. Hence $\bar{x} \in K_\alpha$.

Theorem: Let $E: R^n \to R^n$ be a projection map in R^n and M be a nonempty E-convex subset of R^n. A function $f: R^n \to R$ is a quasi E-convex on M if and only if K_α^E is an E-convex set for each real number α.

Proof: Suppose that f is quasi E-convex, we will get, an E-convexity of K_α^E. Conversely, suppose that K_α^E is an E-convex set for each real number α. For each $x, y \in M$, $\lambda \in [0,1]$ such that $z = \lambda E(x) + (1-\lambda)E(y) \in M$, let $\alpha = \max\{f \circ E(x), f \circ E(y)\}$, thus $x \in K_\alpha^E$, $y \in K_\alpha^E$, since K_α^E is E-convex set, then $z = \lambda E(x) + (1-\lambda)E(y) \in K_\alpha^E \subset K_\alpha$, it follows that $f(z) = f[\lambda E(x) + (1-\lambda)E(y)] \le \alpha = \max\{f \circ E(x), f \circ E(y)\}$

Which shows that f is quasi E-convex.

Theorem: Let $E: R^n \to R^n$ be a projection map in R^n. A subset M of R^n is an E-convex if and only if it contains all the E-convex combinations of its elements.

Proof: Actually, by definition, a set M is an E-convex if and only if $\lambda_1 E(x_1) + \lambda_2 E(x_2) \in M$ whenever $x_1 \in M, x_2 \in M, \lambda_1 \ge 0, \lambda_2 \ge 0$ and $\lambda_1 + \lambda_2 = 1$. In other, the E-convexity of M means that M is closed under taking E-convex combinations with m=2. We must show that this implies M is also closed under taking E-convex combination with m>2. Take any m>2 and make the induction hypothesis that M is closed under taking all E-convex combination of fewer than m vectors. Given an E-convex combination $x = \lambda_1 E(x_1) + \lambda_2 E(x_2) + ... + \lambda_m E(x_m)$ of elements of M, at least one of the scalars λ_i differs from 1; let it be λ_1 for convenience.

Now, put $\lambda_i' = \frac{\lambda_i}{1-\lambda_1}$, we have $\lambda_i' \ge 0$ for i=2, 3, ..., m and

$$\lambda_2' + \lambda_3' + ... + \lambda_m' = \frac{\lambda_2}{1-\lambda_1} + \frac{\lambda_3}{1-\lambda_1} + ... + \frac{\lambda_m}{1-\lambda_1} = 1.$$

Thus $\lambda_2' E(x_2) + \lambda_3' E(x_3) + ... + \lambda_m' E(x_m)$ is E-convex combination of m-1 elements of M and so belongs to M by induction, i.e.

$y = \lambda_2' E(x_2) + \lambda_3' E(x_3) + ... + \lambda_m' E(x_m) \in M$ and so

$$E(y) = \lambda_2' E^2(x_2) + \lambda_3' E^2(x_3) + ... + \lambda_m' E^2(x_m)$$
$$= \lambda_2' E(x_2) + \lambda_3' E(x_3) + ... + \lambda_m' E(x_m).$$

Now, $x = \lambda_1 E(x_1) + (1-\lambda_1)E(y)$, $y, x_1 \in M$ and so $x \in M$.

Theorem: Let M be a nonempty compact polyhedral set in R^n and let $f: R^n \to R$ be quasi E-convex and continuous on M, $E: R^n \to R^n$ be a projection map in R^n and preserve the order of f.

Consider the problem to maximize f(x) subject to $x \in M$. Then an optimal solution $\bar{x}$ to the problem exists, where $\bar{x}$ is an extreme point of M.

Proof: Note that f is continuous on M and, hence, attains a maximum, say at $x' \in M$. If there is an extreme point whose objective is equal to f(x′), then the result is at hand.

Otherwise, let $x_1, x_2, ..., x_k$ be the extreme points of M and assume that $f(x') > f(x_j)$ for j=1, 2, ..., k. By theorem [11], x′ can be represented as $x' = \sum_{j=1}^{k} \lambda_j x_j$, $\sum_{j=1}^{k} \lambda_j = 1$, $\lambda_j \ge 0$, j=1, 2, ..., k.

Since E preserves the order of f, then $f \circ E(x') > f \circ E(x_j)$ for each j, or,

$$f \circ E(x') = f \circ E\left(\sum_{j=1}^{k} \lambda_j x_j\right)$$
$$= f\left(\sum_{j=1}^{k} \lambda_j E(x_j)\right) = f(\bar{x}) > f \circ E(x_j)$$

for each j, where $\bar{x} = \sum_{j=1}^{k} \lambda_j E(x_j)$.

Hence $f(\bar{x}) > \max_{1\le j\le k} f \circ E(x_j) = \alpha$ (2)

Now consider the set $K_\alpha^E = \{x : f \circ E(x) \le \alpha\}$. Note that $x_j \in K_\alpha^E$ for j=1, 2, ..., k and by the quasi E-convexity of f, K_α^E is E-convex. Hence $\bar{x} = \sum_{j=1}^{k} \lambda_j E(x_j)$ belongs to K_α^E and so $f \circ E(\bar{x}) \le \alpha$ (3)

But $f(\bar{x}) \le f \circ E(\bar{x})$ (4)

because if $f(\bar{x}) > f \circ E(\bar{x})$, then we have $f \circ E(\bar{x}) > f \circ E(\bar{x})$ which is impossible. From (3) and (4) we get $f(\bar{x}) \le f \circ E(\bar{x}) \le \alpha$, which contradicts (2). This contradiction shows that $f(x') = f(x_j)$ for some extreme point x_j and the proof is complete.

Quasi E-Convex Bilevel Programming Problem: We assume that the common constraint region to both level in (1) is a polyhedron, i.e.

$$S = \{(x_1, x_2) : A^1 x_1 + A^2 x_2 \le b,\ x_1 \ge 0,\ x_2 \ge 0\} \quad (5)$$

Where, A^1 is an $m x n_1$ matrix, A^2 is an $m x n_2$ matrix and b is an m-vector.
We also assume that S is a nonempty and bounded, so it is a compact polyhedron, which from now on will be called the feasible region.
Moreover, we assume that f_1 is a quasi E-convex and continuous function and f_2 is a quasi E-convex and continuous function, given x_1. Where $E : R^n \to R^n$ be a projection map in R^n and preserve the order of f_1 and f_2.
Finally, in order to assure that the bilevel programming problem is well posed [3, 6], we assume that, for each value of the first-level variables x_1, there will be a unique solution x_2 to the second-level problem.
Throughout the remainder of this paper, we restrict our attention to the bilevel programming problem (1) with the preceding assumptions. From now on, this problem will be called the quasi E-convex bilevel programming (QECBP) problem.
Recall that a face of a convex set T is a convex subset T′ of T such that every closed line segment in T with a relative interior point in T′ has both end points in T′. Since S is a polyhedron, it has a finite number of faces [12]. Let $S_1, \ldots, S_r$ denote the non empty faces of S. Moreover, we will denote by S_{x_1} (S_{x_2}) the projection of S onto R^{n_1} (R^{n_2}). Notice that S_{x_1}, S_{x_2} and S_j, $j \in \{1, \ldots, r\}$ are nonempty compact polyhedra.
Let the point-to-set map Ω from S_{x_1} to S_{x_2} be defined as:

$$\Omega(x_1) = \{x_2 \in R^{n_2} : A^2 x_2 \le b - A^1 x_1,\ x_2 \ge 0\}.$$

Notice that, for each $x_1 \in S_{x_1}$, $\Omega(x_1)$ is a nonempty compact polyhedron, which in fact is the feasible region of the second-level decision maker.
Let the point-to-set map Φ from Sx_1 to Sx_2 be defined as:

$$\Phi(x_1) = \left\{ \begin{array}{l} x_2 \in \Omega(x_1) : f_2(x_1, x_2) \\ = \max\{f_2(x_1, y_2) : y_2 \in \Omega(x_1)\} \end{array} \right\}.$$

Note that, due to the uniqueness of the solution to the second-level problem, $\Phi(x_1)$ is a single-valued at x_1, for each $x_1 \in S_{x_1}$.
The feasible region of the first-level decision maker, called inducible or induced region, will be denoted by:

$$IR = \{(x_1, x_2) : x_1 \in S_{x_1},\ x_2 = \Phi(x_1)\}$$
$$= \left\{ \begin{array}{l} (x_1, x_2) : x_1 \ge 0,\ x_2 \\ = \arg\max\{f_2(x_1, y_2) : A^1 x_1 + A^2 y_2 \le b,\ y_2 \ge 0\} \end{array} \right\}$$

The following lemmas give some properties on the geometry of the feasibility region of the first-level decision maker. In fact these lemmas allow us to show that the inducible region of the QECBP problem is comprised of the union of connected faces of S.

Lemma: The inducible region of the QECBP problem lies on the boundary of S.

Proof: As noted above, for each $x_1 \in S_{x_1}$, the resulting feasible region $\Omega(x_1)$ to the second-level problem is a nonempty compact polyhedron. Taking into account that f_2 is a quasi E-convex and continuous function on $\Omega(x_1)$, where, $E : R^n \to R^n$ be a projection in R^n and preserve the order of f_2, an optimal solution to the second-level problem,

$\max f_2(x_1, x_2)$
subject to $x_2 \in \Omega(x_1)$

exists and occurs at an extreme point of $\Omega(x_1)$. Hence, $\Phi(x_1)$ is an extreme point of the polyhedron $\Omega(x_1)$. Therefore, $\Phi(x_1) \in \partial\Omega(x_1)$, where $\partial\Omega(x_1)$ denotes the boundary of $\Omega(x_1)$, $(x_1, \Phi(x_1)) \in \partial S$ and so the proof is complete.
As a consequence of the previous lemma and taking into account that the collection of all relative interiors of nonempty faces of S is a partition of S [12], for each $(x_1, \Phi(x_1)) \in IR$, there exists a face $S_j \ne S$, such that

$(x_1, \Phi(x_1)) \in riS_j$, where riS_j denotes the relative interior of S_j.
The proofs of the following lemmas appear in [6].

Lemma: The inducible region of the QECBP problem is continuous.

Lemma: Let $(x_1^\circ, \Phi(x_1^\circ)) \in riS_j \cap IR$. Then $S_j \cap V = \{(x_1^\circ, \Phi(x_1^\circ))\}$, where $V = \{(x_1^\circ, x_2) : x_2 \in \Omega(x_1^\circ)\}$.

Lemma: Let S_j be a nonempty face of S and let $(x_1^\circ, \Phi(x_1^\circ)) \in IR$. If $(x_1^\circ, \Phi(x_1^\circ)) \in riS_j$, then $S_j \subset IR$.

Lemma: The inducible region of the QECBP problem is piecewise linear. As we see in the previous lemmas, the inducible region of the QECBP problem is comprised of the union of connected faces of S. This allows us to prove the main result of the paper regarding the optimal solution to the QECBP problem.

Theorem: There is an extreme point of the feasible region S which is an optimal solution to the QECBP problem.

Proof: The QECBP problem can be written equivalently as:

$$\max f_1(x_1, x_2)$$
$$\text{subject to } (x_1, x_2) \in IR$$

Where, $IR = \bigcup_{j \in J} S_j, J \subset \{1, ..., r\}$ and S_j is a face of S. Firstly, notice that the first-level decision maker maximizes a continuous function over a compact set. Hence, there exists a maximizing solution to the QECBP problem [11]. Let this be $(x_1^\circ, \Phi(x_1^\circ))$. Then, there exists at least one $j \in J$ such that $(x_1^\circ, \Phi(x_1^\circ)) \in S_j$ and $(x_1^\circ, \Phi(x_1^\circ))$ is a maximizing solution to the problem.

$$\max f_1(x_1, x_2) \quad (3a)$$
$$\text{subject to } (x_1, x_2) \in S_j \quad (3b)$$

Since f_1 is a quasi E-convex and continuous function on S_j, where $E : R^n \to R^n$ be a projection map in R^n and preserve the order of f_1 and S_j is a nonempty compact polyhedron, it can be concluded that there exists an extreme point of S_j (therefore an extreme point of S) which is an optimal solution to (3), thus giving the same value of the objective function as $(x_1^\circ, \Phi(x_1^\circ))$. Therefore, this extreme point of S is an optimal solution to QECBP problem and the proof is complete.

CONCLUSION

In this study we proved some properties of E-convex sets and quasi E-convex functions and discuss the solution of quasi E-convex bilevel programming problem. This problem assumes that the objective functions of both levels are quasi E-convex, where E is a projection map in R^n preserves the order of f_1 and f_2 and the feasible region is a polyhedron. For this problem, we have proved that it is possible to extend the result concerning the linear bilevel problem which assures that there is an extreme point of the feasible region that solves the problem. This property allows us to consider enumerative methods of searching for extreme points in order to solve the problem.

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