

On Permutable Subgroups of n-ary Groups

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Abstract: It is proved that every permutable subgroup of a finite n-ary group is subnormal.

Key words: finite n-ary group, permutable n-ary group, subnormal n-ary group

INTRODUCTION

We remind that, the system $G = \langle X, () \rangle$ with one n-ary operation $()$ is called n-ary group^[1,2], if it is associative and every one of the equations.

$$(a_1 a_2 \dots a_{i-1} x a_{i+1} \dots a_n) = a$$

is solvable in X, where $a_1, \dots, a_n, a \in X, i = 1, 2, \dots, n$. Throughout this study all n-ary groups are finite. Let G be n-ary group and let H is a subgroup of G, then H is called permutable n-ary group if $HT=TH$ for all subgroups T of G.

It is known however, that every permutable subgroup of a finite group is subnormal^[3,4]. In this study we prove this property for n-ary groups.

PRELIMINARIES

Notation is standard^[2]

X_m^k -the sequence $X_m X_{m+1} \dots X_k$ (if $m = k$ then $X_m^m = x_m$).

Definition 1: Let G be n-ary group, then $x_1^{k(n-1)}$ is an identity if $(x x_1^{k(n-1)}) = (x_1^{k(n-1)} x) = x$ for all $x \in G$.

Definition 2: Let G be n-ary group and let $x \in G$, then the sequence of elements \bar{x} of G is called an inverse of x if $x\bar{x}$ is an identity.

Let $H \leq G$ and x_i^i, y_j^j are sequences of elements of G, where $i + j = k(n-1)$ [$k \in \mathbb{N}$], then the symbol $[X_i^i H y_j^j]$ denote all elements $(x_i^i h y_j^j)$ where $h \in H$.

By analogous of binary groups n-ary subgroup H of a group G is called normal if for any $x \in G$ and for any sequence \bar{x} we have $xH\bar{x} = H$.

Definition 3: N-ary subgroup H of a group G is called subnormal in G if:

$$H = N_0 \leq N_1 \leq \dots \leq N_{t-1} \leq N_t = G$$

Where N_i is a normal in $N_{i+1}, i = 0, 1, \dots, t-1$.

If H and T are subgroups of n-ary group G, then $[H T]$ is the set of all products $(h_1 \dots h_i t_1 \dots t_{n-i})$, where $h_i \in H$ and $t_j \in T$.

Lemma 1^[2]: Let H and T are subgroups of n-ary group G such that

$$\begin{bmatrix} H & T \\ T & H \end{bmatrix} = \begin{bmatrix} T & H \\ H & T \end{bmatrix}, \text{ then } B = \begin{bmatrix} H & T \\ H & T \end{bmatrix} \text{ is a subgroup of } G \text{ and } B \supseteq H.$$

Subgroup H of n-ary group G is called permutable if for any subgroup T from G we have

$$T \cap H \neq \Phi \text{ and } \begin{bmatrix} H & T \\ H & T \end{bmatrix} = \begin{bmatrix} T & H \\ T & H \end{bmatrix}$$

Lemma 2^[2]: Let H and T are subgroups of n-ary group G. If $H \cap T \neq \phi$, then

$$\left| \begin{bmatrix} H & T \\ H & T \end{bmatrix} \right| = \frac{|H||T|}{|H \cap T|}$$

MAIN RESULTS

We are now to prove the following.

Lemma 3: If H u T are permutable subgroups of n-ary group G, then $\begin{bmatrix} H & T \\ H & T \end{bmatrix}$ is permutable subgroup of G.

Proof: Let D any subgroup of n-ary group G, then by the definition of permutable subgroup $H \cap D \neq \phi$. By lemma 1 $H \leq \begin{bmatrix} H & T \\ H & T \end{bmatrix}$ and it is mean that $H \cap D \subseteq$

$$\begin{bmatrix} H & T \\ H & T \end{bmatrix} \cap D \neq \phi.$$

Now since

$$\begin{aligned} \left[\begin{bmatrix} H & T \\ H & T \end{bmatrix} D \right] &= \begin{bmatrix} H & T & T & D^{n-1} \end{bmatrix} = \begin{bmatrix} H & T & D & T \end{bmatrix} = \\ \dots \left[H D T \right] &= \begin{bmatrix} D & H & T \end{bmatrix} = \begin{bmatrix} D & H & T \end{bmatrix} \end{aligned}$$

So $\begin{bmatrix} H & T \\ H & T \end{bmatrix}$ is permutable subgroup in G.

Lemma 4: Let H be a subgroup of n-ary group G. Then if for some element $x \in G$ and for some sequence of inverse (\bar{x}) of x we have $[H H_1^{n-1}] = G$, where $H_1 = xH\bar{x}$, then $H = H_1$.

Proof: Let $x = (a b_1 \dots b_{n-1})$ where $a \in H$ and $b_i \in H_i$. Let \bar{b}_i be a sequence of elements from H_i which are inverses for b_i , $i = 1, 2, \dots, n-1$. Then

$$a = (ab_1 \dots b_{n-1} b_{n-1} \dots b_1) = (x \bar{b}_{n-1} \dots \bar{b}_1).$$

It is clear, that $b_1 \dots b_{n-1} x$ is the sequence of inverses for a . That means if \bar{a} is any sequence of elements of H that inverse for a , then $H = [aH\bar{a}] = [(x\bar{b}_{n-1}\dots b_1)H(b_1 \dots b_{n-1} \bar{x})] = [xH\bar{x}] = H_1$.

Lemma 5: Let x be an element of n -ary group G and let $\varphi_x : G \rightarrow G$ a map defined by $\varphi_x(g) = xg\bar{x}$ where $g \in G$ and \bar{x} is some sequence that is inverse for x . Then φ_x is an automorphism of G .

Proof: For any sequence of element g_1^n from G we have

$$\begin{aligned} \varphi_x(g_1 \dots g_n) &= x(g_1 \dots g_n)\bar{x} = \\ (x(g_1\bar{x})(g_2\bar{x}) \dots (g_{n-1}\bar{x})g_n\bar{x}) &= \\ ((xg_1\bar{x}))(xg_2\bar{x}) \dots (xg_n\bar{x}) &= (g_1^{\varphi_x} g_2^{\varphi_x} \dots g_n^{\varphi_x}) \end{aligned}$$

So φ_x is an endomorphism of n -ary group G .

If $g \in G$, then $\varphi_x(\bar{x}g x) = (x(\bar{x}g x)\bar{x}) = g$. It means that φ_x is an epimorphism. It is obvious that φ_x is an injection.

Theorem: If H is a subgroup of n -ary group G that is permutable with any subgroup of G , then H is a subnormal in G .

Proof: We prove by induction on the order of n -ary group G . let N is the greatest permutable subgroup of G ($N \neq G$) that contains the subgroup H .

We show that N is a normal subgroup of G . let N is not normal subgroup. By the definition of normal subgroup we can find some $x \in G$ such that $xN\bar{x} \neq N$ where **Error! Bookmark not defined.** is some sequence that is inverse of x . let $\varphi_x : G \rightarrow G$ defined by $\varphi_x(g) = xg\bar{x}$ for all $g \in G$. by lemma 5, φ_x is an automorphism n -ary group G . That means $xN\bar{x}$ is a permutable subgroup of n -ary group G u $|N| = |xN\bar{x}|$.

Applying lemma 1 we have $D = \begin{bmatrix} n-1 \\ N N \end{bmatrix} = \begin{bmatrix} n-1 \\ N N \end{bmatrix}$ which contains N subgroup n -ary group G , where $N_1 = xN\bar{x}$. According to lemma 2 the order of this subgroup is:

$$d = \begin{vmatrix} n-1 \\ N_1 N \end{vmatrix} = \frac{|N_1||N|}{|N_1 \cap N|}$$

Since $N \neq xN\bar{x}$ and $|N| = |xN\bar{x}|$, then $d > N$. But by Lemma 3 subgroup D is permutable in G . That means $D = G$ and this contradict lemma 4. So N is a normal subgroup of g . Since $|N| < |G|$ and H is permutable subgroup of N , then by, choosing group G we can conclude that H is subnormal subgroup in N . It means H is a subnormal subgroup of G .

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