

## Conservation of “Partial Vorticity” with Application on Hydraulic Jumps

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**Abstract:** In two preceding works, the idea of partial conservation theorems was introduced and conservation equations for partial energy and partial angular momentum were established. A similar conservation theorem for partial vorticity of incompressible fluids is established here. The vorticity vector is divided into two elements (both denoted “partial vorticity”) and their conservation equations established separately. They show that, in addition to terms similar to the terms of the conservation equation for total vorticity, the conservation equation for partial vorticity has a term that describes the transfer of vorticity between the partial vorticities, i.e. Without affecting the vorticity vector. A simple example of an application is included. It shows that the vortex in the vicinity of a hydraulic jump is located above the surface level of the incoming flow.

**Key words:** Vorticity, conservation theorem, fluid dynamics, hydraulic jump

### INTRODUCTION

In two preceding works<sup>[1,2]</sup>, conservation theorems of partial energy and partial angular momentum were published. In these works, the energy and the angular momentum were split into several parts and separate conservation equations were established for each part. This separation was achieved by realizing that e.g. Energy-even if it is a scalar-is often based on vectors, such as velocity and gravity vectors. In order to make the direction information of these vectors available, partial energy was established by a similar procedure as total energy, but based on one component of the equation of motion at a time instead of the complete equation. A similar procedure was adopted to establish partial angular momentum. The latter provides a means to separate angular momentum of the waves (denoted “wave spins”<sup>[3,4]</sup>), from the angular momentum of horizontal currents<sup>[5,6]</sup>. By the conservation laws it was possible to study the interaction between waves and currents and even consequences of wave breaking. The latter was possibly because conservation equations can be based on the situation before and after wave breaking, without having to treat the rather chaotic processes of wave breaking in the meantime<sup>[5]</sup>.

In order to complete the set of partial conservation equations applicable to fluid dynamics, the conservation equation of partial vorticity is established here. (See e.g.<sup>[7]</sup> for vorticity). As in the two previous cases, they include terms similar to the conservation equation of total vorticity, but also transfer terms that describe to what extent vorticity is transferred between the partial vorticities. As an example of application, the flow in a hydraulic jump is briefly studied.

**Basic conservation equations:** In order to split a vorticity vector into two elements and to establish the transfer terms that determine the transfer of vorticity between the elements, the partial vorticity is defined as:

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_j} \quad (1)$$

Here  $i$  and  $j$  can have the values 1, 2 and 3, but not the same value. Further  $u_i$  is the velocity component in the direction of the  $i^{\text{th}}$  axis of a Cartesian coordinate system and  $x_i$  the corresponding coordinates.

Partial vorticity is established from the  $i$ -th component of the equation of motion of an incompressible, viscous fluid:

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x_i} + u_j \frac{\partial u_i}{\partial x_j} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i - \delta_{i3} g \quad (2)$$

where,  $t$  is the time,  $\rho$  the density of the fluid,  $\nu$  its kinematic viscosity,  $p$  the pressure and  $\delta$  the Kronecker delta. Further, the acceleration of gravity ( $g$ ) is assumed to be in the negative direction of the  $x_3$  axis. The summation rule is not adopted.

In order to obtain the conservation equation for partial vorticity, differentiation with respect to  $x_j$  is performed:

$$\begin{aligned} \frac{\partial^2 u_i}{\partial t \partial x_j} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial^2 u_i}{\partial x_i \partial x_j} + \frac{\partial u_k}{\partial x_j} \frac{\partial u_i}{\partial x_k} = \\ - u_k \frac{\partial^2 u_i}{\partial x_k \partial x_j} - \frac{\partial u_j}{\partial x_j} \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial^2 u_i}{\partial x_j^2} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \nabla^2 \frac{\partial u_i}{\partial x_j} \end{aligned} \quad (3)$$

By reorganizing and adding and subtracting  $\partial u_k / \partial x_k$  in the parenthesis:

$$\frac{\partial^2 u_i}{\partial t \partial x_j} + \frac{\partial u_i}{\partial x_j} \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j} + \frac{\partial u_k}{\partial x_k} - \frac{\partial u_k}{\partial x_k} \right) + u_i \frac{\partial^2 u_i}{\partial x_i \partial x_j} + \frac{\partial u_k}{\partial x_j} \frac{\partial u_i}{\partial x_k} =$$

$$- u_k \frac{\partial^2 u_i}{\partial x_k \partial x_j} - u_j \frac{\partial^2 u_i}{\partial x_j^2} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_j} + v \nabla^2 \frac{\partial u_i}{\partial x_j} \quad (4)$$

The sum of the first three terms in the parenthesis equals  $\text{div } v$ , which vanishes for an incompressible fluid. So, for an incompressible fluid, after shuffling the terms:

$$\frac{\partial^2 u_i}{\partial t \partial x_j} + u_i \frac{\partial^2 u_i}{\partial x_i \partial x_j} + u_k \frac{\partial^2 u_i}{\partial x_k \partial x_j} + u_j \frac{\partial^2 u_i}{\partial x_j^2} =$$

$$\frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_k} - \frac{\partial u_k}{\partial x_j} \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_j} + v \nabla^2 \frac{\partial u_i}{\partial x_j} \quad (5)$$

The left hand side is the total derivative of the partial vorticity. Hence:

$$\dot{\gamma}_{ij} = \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_k} - \frac{\partial u_k}{\partial x_j} \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_j} + v \nabla^2 \gamma_{ij} \quad (6)$$

Here the dots above  $\gamma_{ij}$  denotes total time differentiation.

The two first terms at the right hand side are functions of  $u_i, u_j$  and  $u_k$ . These terms vanish for a two-dimensional flow, so that:

$$\dot{\gamma}_{ij} = v \nabla^2 \gamma_{ij} - \theta_{ij} \quad (7)$$

Where:

$$\theta_{ij} = \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_j} \quad (8)$$

In Eq. (7),  $i$  and  $j$  may exchanged. Then, since  $\theta_{ij} = \theta_{ji}$

$$\dot{\gamma}_{ji} = v \nabla^2 \gamma_{ji} - \theta_{ij} \quad (9)$$

The  $k^{\text{th}}$  component of the vorticity vector ( $V_k$ ) can be written as a difference between the two partial vorticities:

$$V_k = \gamma_{ij} - \gamma_{ji} \quad (10)$$

Hence by subtracting Eq. (9) from Eq. (7), the conventional vorticity equation in two dimensions is obtained. If the two  $\theta_{ij}$  terms are non-zero, vorticity is transferred between the two partial vorticities i.e. Without affecting the total vorticity of the system. It

implies that  $\theta_{ij}$  works as a transfer term for vorticity between the two partial vorticities,  $\gamma_{ij}$  and  $\gamma_{ji}$ . So by establishing the equation for partial vorticity, a new term is established that gives additional information regarding the flow that is not obtained from the ordinary vorticity equation.

**Conservation equations for a two-dimensional volume:** Consider a two-dimensional flow in the  $x$ - $y$  plane that is not a function of the third coordinate. If viscosity is ignored and Eq. (7) is integrated over an area  $A$  in the  $x$ - $y$  plane surrounded by a closed curve  $C$ , then:

$$\dot{\Gamma}_{xy} = - \frac{1}{\rho_A} \int \frac{\partial^2 p}{\partial x \partial y} dx dy \quad (11)$$

where,  $\Gamma_{xy}$  is the integral of the partial vorticity based on the  $x$ -component of the fluid velocity  $u$ , i.e.:

$$\Gamma_{xy} = \int_A \frac{\partial u}{\partial y} dx dy \quad (12)$$

Green's theorem is given as<sup>[8]</sup>:

$$\int_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_C (P dx + Q dy) \quad (13)$$

where the line integral is taken in the counter-clockwise direction. By choosing  $P = 0$  and **Error! Bookmark not defined.**, eqn. (13) gives:

$$\int_A \frac{\partial^2 p}{\partial x \partial y} dx dy = \oint_C \frac{\partial p}{\partial y} dy \quad (14)$$

Similarly, by choosing  $P = -\partial p / \partial x$  and  $Q = 0$  in Greens theorem:

$$\int_A \frac{\partial^2 p}{\partial x \partial y} dx dy = - \oint_C \frac{\partial p}{\partial x} dx \quad (15)$$

Hence we may write Eq. (11) either as

$$\dot{\Gamma}_{xy} = \frac{1}{\rho} \oint_C \frac{\partial p}{\partial x} dx \quad (16)$$

or as:

$$\dot{\Gamma}_{xy} = - \frac{1}{\rho} \oint_C \frac{\partial p}{\partial y} dy \quad (17)$$

Under special circumstances these integrals are particularly simple: In Eq. (16), since  $dx = 0$  along the parts of  $C$  that are parallel to the  $y$  axis, the integral

vanishes along these lines. Similarly if parts of  $C$  are parallel to the  $x$  axis they do not contribute in Eq. (17). Further, if  $C$  is following a line where  $p = \text{constant}$  (e.g. a free surface) both integrals above vanish.

The equation is even simpler if the area  $A$  is rectangular and the sides of the rectangle are parallel to the coordinate axes. In this case, let the lower left corner be at  $x_1, y_1$  and the upper right corner at  $x_2, y_2$ . Then integration of either Eq. (11), (16) or (17) all imply:

$$\dot{\Gamma}_{xy} = \frac{1}{\rho}(p_{11} + p_{22} - p_{12} - p_{21}) \quad (18)$$

So, in the non-viscous, two-dimensional case, whatever happens inside the volume, the transfer of vorticity between a pair of partial vorticities depend on the pressure of the four corners of the rectangle only.

**Application of partial vorticity on a hydraulic jump:**

The hydraulic jump is theoretically treated in many textbooks, e.g.<sup>[9]</sup>. The conventional theory gives little information on the velocity distribution inside the jump, however, as only mean velocities over the depth are considered. By the partial vorticity it is possible to obtain more detailed information. Hence the hydraulic jump is studied in the following.

Figure 1 shows a cross section of a hydraulic jump, where the flow is coming from the left as indicated by an arrow. The five points shown on the figure form two vertical lines. They are located where the flow can be considered horizontal and the pressure hydrostatic, but near enough the jump to disregard the consequences of viscosity in the flow below the mutual level of point 2 and point 5.

First, the rectangle 1 – 2 – 5 – 4 is treated. For this rectangle, Eq. (18) can be written as:

$$\dot{\Gamma}_{xy} = \frac{1}{\rho}(p_4 + p_2 - p_5 - p_1) \quad (19)$$

where the subscripts refer to the numbers of the points in Fig. 1. Since the pressure is static at both ends of the control volume,  $p_4 - p_5 = p_1 - p_2$  and therefore the contents of the parenthesis in eqn. (19) vanishes. Hence  $\dot{\Gamma}_{xy} = \text{constant}$ . This implies that the net transfer imposed by the non-viscous transfer term vanishes below the surface level of the incoming water. Instead of point 2 and point 5, any pairs of points at a mutual level below them can be chosen with exactly the same result. Consequently, any vorticity of the incoming flow remains unchanged from cross-section 5-4 to cross-section 2 – 1. As viscous forces within the rectangle are neglected, the incoming flow remains unchanged since the three-dimensional terms in Eq. (6) cannot develop by any other means. Consequently the vortex has to be located above the surface level of the incoming flow.

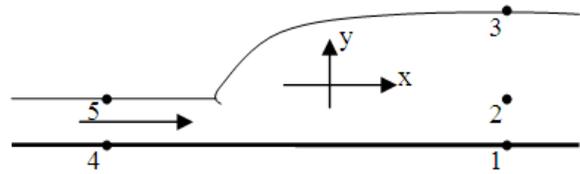


Fig. 1: Definition sketch of a hydraulic jump

Clearly the viscous forces cannot be disregarded over long distances. Hence this conclusion is only valid near the jump.

Another alternative is to treat all fluids below the surface between point 3 and point 5, i.e. The area described by straight lines from 5 to 4 and further to 1 and 3 and back along the free surface to 5. For this case Eq. (17) is adopted. Since  $p$  is constant at the surface and  $dy = 0$  at the bottom, their contribution to the integral vanish. Hence only the two vertical lines contribute. Since the pressure is static at both ends of the volume under consideration:

$$\dot{\Gamma}_{xy} = -g \Delta H \quad (20)$$

where  $\Delta H$  is the increase of the surface level through the hydraulic jump. Further, Eq. (10) implies that

$$\dot{\Gamma}_{xy} = \dot{\Gamma}_{yx}, \text{ where:}$$

$$\dot{\Gamma}_{yx} = \int_A \frac{\partial v}{\partial x} dx dy \quad (21)$$

in which  $v$  is the velocity component in the  $y$  direction. Therefore Eq. (20) implies that also:

$$\dot{\Gamma}_{yx} = -g \Delta H \quad (22)$$

As a consequence, both partial vorticities are generated by the transfer term in such a way that the total vorticity remains unchanged. Since  $\partial u / \partial y$  and  $\partial v / \partial x$  both are negative, the well known vortex is allowed. The first term allows a negative horizontal surface flow to develop downstream the jump, while the second term feeds vertical flows. According to the first part of this section, the vortex is basically located above the surface level of the incoming flow. Hence the flow field in the vicinity behind the hydraulic jump is to some extent explained by the non-viscous terms.

Any further discussion of the flow in the vortex must include all terms of Eq. (6) since the flow is strongly turbulent here. As the purpose of this example is to show how the conservation equations of partial vorticity work, a further discussion of the hydraulic jump is considered to be outside the scope of this study.

## CONCLUSION

In three studies, one of partial energy<sup>[1]</sup>, one with partial angular momentum<sup>[2]</sup>, and this on partial vorticity, three partial conservation quantities for fluid dynamics have been established. They form a new set of conservation equations for use in fluid dynamics. By establishing conservation equations based on only one component of the equation of motion at a time, it is possible to study the interaction between different flow regimes by simple means. This has been demonstrated briefly here, where the horizontal flow and the vertical flow of a hydraulic jump are considered and their interactions-and absence of interactions-have been studied. For this purpose the transfer term between the two partial vorticities is described in Eq. (8) and on integral form of a two-dimensional non-viscous flow in Eq. (16)-(18).

Whether energy, angular momentum or vorticity is treated, all three types of partial conservation equations have terms that give information on the transfer between e.g. Horizontal and vertical flows. From these transfer terms, information can be obtained on the stability of the flow, as non-zero transfer terms imply a flow that changes, either in time or space. As demonstrated, these conservation equations open for the solution of new problems and further insight into other problems. Here we have seen that the vortex in the vicinity of a hydraulic jump is located above the surface level of the incoming flow. By adopting partial angular momentum equations on water waves, based on the vertical flow only, the horizontal currents completely disappeared from the equations<sup>[2]</sup>. Hence disruptive effects of waves-separated from horizontal currents-could be studied, with unexpected consequences as down shifting induced by dissipation as a result. In general, the partial equations open for studies of interactions between different flow regimes without having to adopt detailed numerical approaches. On the other hand, Eq. (18) appears well suited for numerical applications, as it is based on the pressure at the corners of rectangular control areas.

This study, as<sup>[1,2]</sup> are first and foremost written to establish the partial conservation concept. Hence the application parts are merely meant as examples. Probably better applications exist and it is the author's belief that the partial conservation equations will turn out to be valuable in the future. I leave that as a challenge to the reader.

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