American Journal of Applied Sciences 1 (4): 258-260, 2004
ISSN 1546-9239


# Fourier Series With Spectral and Wave Number Decomposition

Jung-Hua Chou and Raylin Wu
Department of Engineering Science,
National Cheng Kung University Tainan, Taiwan, 701

**Abstract:** In this study, the generic nature of Fourier series is pointed out and the requirement of dimensional homogeneity for physical systems is emphasized. By suitable application of non-dimensional variables, both time versus frequency and spatial versus wave number decomposition of Fourier series are included properly to put the physics involved into the right physical meanings. Furthermore, the high frequency noise features through differentiation and integration are revealed nicely by the Fourier series representation of a signal including noise.

**Key words:** Fourier Series, Time Versus Frequency, Spatial Versus Wave Number Decomposition

## INTRODUCTION

Fourier series and related transforms have been used quite extensively in engineering application for their ability to reveal the spectral contents of a signal. It can be easily shown that the Sturm-Liouville equation $\frac{d^2y}{dx^2}+\beta^2 y=0$, with periodic bone (air conditions y (-L) = y(L) and $\frac{dy(-L)}{dx}=\frac{dy(L)}{dx}$ for a period of 2L, has the eigenvalues of $\beta=\frac{n\pi}{L}, n=0,1,2,$ etc. Its solution is simple:

$$y=\sum_{n=0}^{\infty}(a_n\cos\frac{n\pi x}{L}+b\sin\frac{n\pi x}{L})$$

This is a typical representation of Fourier series. Because of its periodic nature, Fourier series is particularly applicable to wave related phenomena. Conventionally, Fourier series for a periodic function f (x) with a period of 2L is typically written as:

$$f(x)=\sum_{n=0}^{\infty}(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}) \quad (1)$$

The coefficients $a_n$ and $b_n$ can be obtained from Euler formulas.

Equation (1) can be recast into its spectral components by treating $2\pi/2L = \omega$ as the fundamental angular frequency of f (x) as follows:

$$f(x)=\sum_{n=0}^{\infty}(a_n\cos n\omega x+b_n\sin n\omega x) \quad (2)$$

The corresponding Euler formulas can be expressed by:

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)$$
$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos n\omega x dx \quad (3)$$
$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin n\omega x dx$$

If $\omega_n=n(\frac{2\pi}{2L})=n\omega, n=1,2,3...$ is used in equation (2), then the spectral decomposition of f(x) can be expressed by the fundamental frequency ω and its harmonics, in addition to the dc component $a_0$. The argument given above as typically given in Advanced Engineering Mathematics textbooks, e.g.,[1-3], is fine if the dimension or unit of the independent variable x is of no concern; namely, the variable x is treated as a generic variable and can be used to represent either a time or a spatial coordinate. On the other hand, physical phenomena always require a proper dimension or unit for which dimensional homogeneity is preserved. Therefore, in all of the above equations, if the variable x is treated as time t, then the spectral components are properly expressed and the period of 2L takes the dimension of $t^{-1}$. However, if the variable x is considered as a spatial coordinate, then 2L has to assume the dimension of $x^{-1}$ so that the dimensional consistency is insured.

The requirement of a dimensionless argument for either a sine or cosine function can be readily seen from the Taylor series expansion for these functions.

For example, the Taylor series expansion of the cosine function cos x about x = 0 is:

$$\cos x=1-\frac{x^2}{2}+\frac{x^4}{24}-... \quad (4)$$

It can be seen that the first, second and third term on the right hand side of equation (4) are 1, $x^2/2$ and $x^4/24$, respectively. The first term is a pure number and thus dimensionless. The second and third term has the dimensions of x to the power of 2 and 4, respectively. If x is not dimensionless, dimensional inhomogeneity is implicitly hidden in these functions and can lead to inconsistent results. Therefore, the importance of dimensional homogeneity of physical systems has to be stressed to avoid the wrong application of these types of functions. In fact, this concept applies equally well to Laplace transform and any other integral transformations.

For the Fourier series in spatial coordinates, the concept of the fundamental frequency is not suitable. Instead, a dimension parallel to frequency and yet pertinent to the spatial coordinate has to be used for dimensional homogeneity.

**Fourier decomposition in wave number space:** The fascinating color exhibited by animals, birds, insects and plants has attracted investigations of many scientists. In general, living systems have two ways of making their colors, namely, pigments and structural colors. The principles of colorization are quite different for them. Pigment relies on the differential absorption of different wavelengths; whereas structural colorization is based on interference or diffraction through physical structures. Because of this difference, the colorization by physical structures tends to last longer and can be seen under low light levels, such as colors observed from the sea mouse of deep sea[4].

Photonic crystals are the technological counterparts of nature's colorization through physical structures. In photonic crystals, the periodic variation of physical properties, such as dielectric constant, is made by various manufacturing techniques. Because of their unique ability of color confinement, they are considered to be an important emerging technology. Thus, it is essential for Fourier series decomposition of spatial periodic functions applicable to these types of structures. In the situation of x being a spatial coordinate, there are several candidates for the period 2L. Among them are the wavelength $\lambda$ of a wave, the lattice constant of an atomic structure and the spatial periodicity of the phenomenon considered as in material dielectric periodicity of photonic crystals, just to name a few examples.

Taking $2L = \lambda$ as an example, the fundamental angular frequency w needs to be replaced by a corresponding wave number $k = \frac{2\pi}{2L} = \frac{2\pi}{\lambda}$ Thus, Fourier series given above becomes:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nkx + b_n \sin nkx)$$
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \qquad (5)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos nkxdx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin nkxdx \qquad (6)$$

Comparing equations (5) and (6) to (2) and (3), it is easy to see that they are identical except the difference in notations. Since mathematicians always seek generic expression for the variables to take care of different situations in different fields, it is reasonable that the Fourier series of the textbooks on Advanced Engineering Mathematics for engineering majors is usually given by equation (1), e.g.,[1-3]. On the other hand, as the technology evolves continuously, situations with spatial periodicity will become a commonality in the future, especially in the fields of optoelectronics, in addition to the conventional field of image processing and material science. Thus, both spectral and spatial decomposition of Fourier series are equally important, especially the associated physical meanings.

It should be mentioned that by expressing Fourier series in either form of equation (2) or (5), the coefficients obtained from Euler formulas assume the dimension of the function f(x). This situation can be achieved only when both sine and cosine functions are dimensionless. That is to say, the arguments of sine and cosine functions have to be dimensionless as stated previously.

The importance of decomposition in both spectral and spatial domain can be further explored from the following wave equation for which Fourier series is a natural solution:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (7)$$

In the above equation, the symbol t, x and c denote the time and spatial coordinate and the speed of wave propagation, respectively. It can be checked easily that the equation is dimensional homogeneous. Its solution in terms of characteristic is[1-3]:

$$u(x,t) = f(x - ct) + g(x + ct) \qquad (8)$$

However, the arguments x-ct and x+ct assume the dimension of a spatial length and are not dimensionless. If equation (8) is to be rephrased into a sine or cosine form, difficulties will arise. On the other hand, if x in equation (7) is made non-dimensional by the wave number k, then equation (7) becomes:

$$\frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{k^2}\frac{\partial^2 u}{\partial x^2} \tag{9}$$

where, the notation $w^2$ represents $k^2c^2$. In other words, by normalizing by x 1/k, the dimensionless variable $\omega t$ appears as a natural consequence. Then its solution in terms of different traveling modes is:

$$u = f(kx - \omega t) + g(kx + \omega t) \tag{10}$$

Comparing to the arguments of both functions of equation (8), the arguments of both functions in equation (10), kx-$\omega$t and kx + $\omega$t, are dimensionless. Thus, equation (10) can be expressed in terms of Fourier series:

$$u = \sum_{n=0}^{\infty} a_n \cos(nkx + n\omega t + \phi_1) + \sum_{n=0}^{\infty} b_n \sin(nkx - n\omega t + \phi_2) \tag{11}$$

where, $\phi_1$ and $\phi_2$ are phase angles. It should be noted that the right traveling mode of the equation (11) is written in sine form to match the general expression of Fourier series. In fact, the right traveling mode can be changed into the cosine form since a phase angle of $\pi/2$ can be easily incorporated into $\phi_2$ without loss of generality. Moreover, the equation (11) can be extended to three-dimensional wave equations readily by replacing k and x by a corresponding wave number vector k and a position vector r, respectively. Namely:

$$u = \sum_{n=0}^{\infty} a_n \cos(n\vec{k}.\vec{r} + n\omega t + \phi_1) + \sum_{n=0}^{\infty} b_n \sin(n\vec{k}.\vec{r} - n\omega t + \phi_2) \tag{12}$$

Thus, both spectral and spatial domains are included properly and dimensional homogeneity is assured by having the dimensions of $a_n$ and $b_n$ the same as that of u.

**Noise features of Fourier series:** The frequency dependence of Fourier series reveals an interesting feature of noise amplification and suppression through differentiation and integration, respectively. Consider a function f(t) composes of a signal $a_s \cos \omega_s t$ and a noise $a_n \cos \omega_n t$ as follows:

$$f(t) = a_s \cos\omega_s t + a_n \cos\omega_n t \tag{13}$$

Through differentiation and integration of equation (13), one can obtain the following equations, respectively:

$$\frac{df(t)}{dt} = -a_s\omega_s \sin\omega_s t - a_n\omega_n \sin\omega_n t \tag{14}$$

$$\int f(t)dt = \frac{a_s}{\omega_s}\sin\omega_s t + \frac{a_n}{\omega_n}\sin\omega_n t + c \tag{15}$$

Thus, if the signal to noise ratio of f (t) is defined as $a_s/a_n$, it can be seen that the signal to ratios become $(a_s/a_n)(\omega_s/\omega_n)$ and $(a_s/a_n)(\omega_n/\omega_s)$ through differentiation and integration, respectively. It is clear that differentiation tends to reduce the signal to noise ration by a factor of $\omega_s/\omega_n$ if $\omega_n > \omega_s$. In other words, differentiation tends to increase the high frequency noise levels. On the other hand, integration tends to reduce the high frequency noise levels.

It needs to point out that a similar conclusion also be drawn if the frequency $\omega$ in equation (13) is replaced by the wave number k. Namely, higher wave number noises tend to make the signal noisier after differentiation and vice versa.

## REFERENCES


1. Kreyszig, E., 1999. Advanced Engineering Mathematics. 8th Ed., John Wiley & Sons.
2. O'Neil, P.V., 1991. Advanced Engineering Mathematics. 3rd Ed., Wadsworth Publishing Company.
3. Greenberg, M.D., 1998. Advanced Engineering Mathematics. 2nd Ed., Prentice-Hall Int. Inc.
4. McPhedran, R.C., N.A. Nicorovici, D.R. McKenzie, L.C. Botten, A.R. Parker and G.W. Rouse, 2001. The sea mouse and photonic crystals. Aust. J. Chem., 54: 1-4.