Variable Separation and Boubaker Polynomial Expansion Scheme for Solving the Neutron Transport Equation

Dada, O.M., O.B. Awojoyogbe, M. Agida and K. Boubaker
Department of Physics,
Federal University of Technology, Minna, Niger State, Nigeria
Ecole Superieure des Sciences et Techniques de Tunis/63,
Rue Sidi Jabeur 5100, Mahdia, Tunisia

Abstract: Problem statement: In this study, we present general analytical solutions to the Neutron Boltzmann Transport Equation NBTE using a polynomial expansion scheme. Approach: Some simple assumptions have been introduced in the main system thanks to the Boubaker Polynomial Expansion Scheme (BPES) in order to make the general analytical procedure simple and adaptable for solving similar real life problems. Results: Finding particular solution to the Neutron equation by making use of boundary conditions and initial conditions may be too much for the present study and reduce the generality of the solutions. Conclusion: The proposed analytical solution of the neutron transport equation has been positively compared to some recently publish results. I should present a relevant supply to studies on reactor modeling.

Key words: Neutron transport equation, distribution function, Boubaker Polynomial Expansion Scheme (BPES), source function, neutron angular flux, analytical solutions, seems appropriate, describing neutron, boltzmann equation

INTRODUCTION

The deterministic neutron transport equation (or Boltzmann neutron transport equation) describes the expected or probable neutron angular density with respect to position, direction, energy and time. Its solutions generally provide average values of the neutron angular density which takes into account random effects from neutron interactions and sources. Since neutrons are neutral particles which possess mass, they can penetrate deep into matter in a nondestructive way despite having characteristics strong forces, a quantum mechanical description seems appropriate, leading to an involved system of Schrödinger equations describing neutron motion between and within nuclei (Kulikowska, 2000; Lewis and Miller, 1984; Bell and Glasstone, 1970; Stammler and Abbate, 1983; Singh, et al., 2010;Nourazar et al., 2011).

From a theoretical point of view, a neutron is a neutral point particle, experiencing deflection from or by a nucleus at the center of an atom. If the conditions are just right, the captured neutron causes a fissile nucleus to fission, producing more neutrons. A stochastic partial differential equation is hence derived from these events of neutron transport in a general three-dimensional absorbing and anisotropic-scattering medium where the neutron angular density depends on position, direction, energy and time (Lewis and Miller, 1984). Sometimes, the nucleus that remains as fission product coincides with one of the stable nuclei. The product nucleus is then different from other nuclei, the reason being that the product nucleus is not stable. It disintegrates further, with a mean life characteristic of the nucleus, by emission of an electric charge, until it finally reaches a stable state.

In the present investigation, the medium is assumed to be constant with respect to material composition, i.e., zero power noise. Special random effects, for example, from randomly varying boundary conditions or from a medium that is randomly varying are not considered in the present investigation although generalizations to approximate such conditions may be possible.

MATERIALS AND METHODS

The neutron transport equation models the transport of neutral particles in a scattering, fission and absorption set of events with no self-interactions (Lewis and Miller, 1984; Chandrasekhar, 1960; Davison, 1957).

In a neutron scattering event (Fig. 1), the quantity to be determined is called the partial differential cross section related to $\Delta \Omega$. This function depends on the energy and momentum transferred from the neutron to the sample. In the most general case the partial differential cross section is a function of four variables since the momentum transfer is a vector quantity with...
three components. In elastic scattering, the neutron does not transfer any energy to the sample. The elastic differential cross section is consequently a function of momentum transfer only. From the number of neutrons counted in a detector, the mean values for variables $\theta$ and $\phi$ can be defined (Fig. 1). $\theta$ is the angle through which the neutron has been scattered, i.e. the angle between the incident and the scattered beam.

The paths of the neutrons which are scattered through $\theta$ form a cone so a second (azimuthal) angle, $\phi$, is also needed to define the detector position. Several analytical and numerical approaches have been used in order to solve the neutron transport equation. Jaffel and Vidal-Madjar (1989); Case and Hazeltine (1970) and Davies (2002) performed discretizing protocols based on Fourier transform, while Cardona and Villhena (1994); Kim and Ishimaru (1999); Kim and Moscoso (2002); Boyd (2001); Bernardi and Maday (1992) and Kadem (2008) used polynomial expansion schemes.

In the actual investigation, the first assumption is that all particles including nuclei are in motion with like particle collisions allowed, as stated by Kulikowska (2000); Mokhtar-Kharroubi (1997) and Kadem (2006). The governing equation are Eq. 1:

$$\int \frac{1}{v} \frac{\partial}{\partial t} + \Omega \nabla + \Sigma(r,E,t) \psi(r,\Omega,E,t) = \int \frac{dE'}{4\pi} \int d\Omega' \Sigma_s(r,\Omega',E' \rightarrow E) \psi(r,\Omega',E',t) + \frac{\chi(E)}{4\pi} \int \frac{dE'}{4\pi} \int d\Omega' \Sigma_s(r,\Omega',E',t) \psi(r,\Omega',E',t) - Q(r,\Omega,E,t)$$

(1)

With:

$V$ = The neutron speed  
$\Sigma, \Sigma_f$ = Macropscopic cross-sections  
$\Sigma_s$ = Scattering cross-section  
$\chi(E)$ = The distribution function  
$\psi$ = The neutron angular flux  
$E,E'$ = Energies  
$\Omega, \Omega'$ = Neutron directions  
$Q(r,\Omega,E,t)$ = The source function (Kulikowska, 2000)

Which give, using variable separation Eq. 2:

$$\frac{\psi}{v} \frac{\partial \psi}{\partial t} + \Omega \nabla \psi + \Sigma(r,E,t) \psi = \psi \Sigma_s(r)$$

$$\int \frac{dE'}{4\pi} \int d\Omega' \Sigma_s(r,\Omega',E' \rightarrow E) \psi' = \frac{\chi(E)}{4\pi} \int \frac{dE'}{4\pi} \int d\Omega' \Sigma_s(r,\Omega',E',t) \psi + Q(r,\Omega,E,t)$$

(2)

Fig. 1: Neutron scattering event geometrical model

which gives Eq. 3:

$$\frac{\psi(t)}{\psi(0)} + \frac{\pi}{4\Omega} \frac{d\psi}{dr} + \Sigma \psi = \psi \Sigma_s$$

$$\int \frac{dE}{4\pi} \int d\Omega \Sigma_s(r,\Omega,E \rightarrow E) \psi' + \frac{\chi(E)}{4\pi} \int \frac{dE}{4\pi} \int d\Omega \Sigma_s(r,\Omega',E',t) \psi + Q(r,\Omega,E,t)$$

(3)

where, $V_E$ represents the average number of neutrons per fission.

By setting Eq. 4:

$$\psi(t) = R(t)G(t)$$

$$A(r,t) = A_1(r) + A_2(t)$$

$$Q_1(t) = \psi Q_0$$

$$Q_0 = Q_1(t) + G_0(t)$$

(4)

One obtains:

$$\int \frac{dE}{4\pi} \int d\Omega \Sigma_s(r,\Omega,E \rightarrow E) \psi' + \frac{\chi(E)}{4\pi} \int \frac{dE}{4\pi} \int d\Omega \Sigma_s(r,\Omega',E',t) \psi + Q(r,\Omega,E,t)$$

(5)
Since both sides of Eq. 5 are independent of one another, they must be equal to a constant \( \varepsilon \); leading to the following equations Eq. 6 and 7:

\[
\frac{dG}{dt} = \frac{Q_0 v}{\Omega} \left[ G_0(t) - A_1(t) \right] G(t) + \varepsilon \tag{6}
\]

\[
\frac{dR}{dr} = \frac{Q_0}{\Omega \psi} \left[ R_0(r) + \varepsilon - \frac{A_1(r)}{\Omega} \right] R(r) - \varepsilon \tag{7}
\]

which have as solutions Eq. 8 and 9:

\[
G(t) = \sum \left\{ \left[ \frac{Q_0}{\Omega \psi} \left[ G_0(t) - A_1(t) \right] \right] G(t) + \varepsilon \right\} dt \tag{8}
\]

\[
R(r) = \sum \left\{ \left[ \frac{Q_0}{\Omega \psi} \left[ R_0(r) + \varepsilon - \frac{A_1(r)}{\Omega} \right] \right] R(r) \right\} dr \tag{9}
\]

where, \( P_0 \) and \( P_1 \) are constants.

The expression for the flux is hence given as Eq. 10:

\[
\frac{1}{\nu} + \frac{\partial}{\partial t} + \nabla \cdot \left[ \sigma_i \psi \right] = \sum \left\{ \sum \left\{ \left[ \frac{Q_0}{\Omega \psi} \left[ G_0(t) - A_1(t) \right] \right] G(t) + \varepsilon \right\} dt \right\} \tag{10}
\]

RESULTS

Different expressions to \( Q_0, G_0, A_2, A_1 \) and \( R_o \) allow obtaining different expressions of the macroscopic cross section which is generally expressed as Eq. 11:

\[
\Sigma_i(r, E, t) = N_i(r, t) \sigma_i(E) \tag{11}
\]

For a given nuclide \( j \) and reaction type \( i \), where \( N_i(r, t) \) is the nuclear atomic density and \( \sigma_i(E) \) is the microscopic cross section, then, if we assume that nuclear atomic density is independent of time Eq. 12 and 13:

\[
\Sigma_i(r, E, t) = N_i(r) \sigma_i(E) \tag{12}
\]

Then:

\[
\begin{align*}
\Sigma_i(t) &= 1 \\
\Sigma_i(r, E) &= \Sigma_i(r) \Sigma_i(E)
\end{align*} \tag{13}
\]

And the expression for the distribution function \( \chi(E) \) becomes (Kulikowska, 2000; Kadem, 2006; Mokhtar-Kharroubbi, 1997) Eq. 14:

\[
\frac{d\chi(E)}{dr} = B(\bar{r}) \psi_i \tag{17}
\]

DISCUSSION

Criticality and analytical solutions can be discussed by considering the special case of the neutron transport equation without delayed neutrons, which is traduced by the equation:

\[
\begin{align*}
\psi_i(\bar{r}, \bar{E}, \bar{t}) &= \psi_i(\bar{r}, \bar{E}, \bar{t}) + \nabla \cdot \left[ \sigma_i \psi_i \right] \\
\frac{1}{\nu} + \frac{\partial}{\partial t} + \nabla \cdot \left[ \sigma_i \psi_i \right] &= \sum \left\{ \sum \left\{ \left[ \frac{Q_0}{\Omega \psi} \left[ G_0(t) - A_1(t) \right] \right] G(t) + \varepsilon \right\} dt \right\}
\end{align*} \tag{14}
\]

With:

\( Q_{ext} \) = The external sources of neutrons  
\( \sigma = \) The microscopic cross-section  
\( v = \) The neutron speed  
\( V_{el} = \) Average number of neutrons per fission

This equation assumes that all neutrons are emitted instantaneously at the time of fission although small fraction of neutrons is emitted later due to certain fission products.

If we look for an asymptotic solution to Eq. 15, satisfying the source free boundary conditions \( Q_{ext} = 0 \), it gives:

\[
\begin{align*}
\frac{1}{\nu} + \frac{\partial}{\partial t} + \nabla \cdot \left[ \sigma_i \psi_i \right] &= \sum \left\{ \sum \left\{ \left[ \frac{Q_0}{\Omega \psi} \left[ G_0(t) - A_1(t) \right] \right] G(t) + \varepsilon \right\} dt \right\} \\
\frac{1}{\nu} + \frac{\partial}{\partial t} + \nabla \cdot \left[ \sigma_i \psi_i \right] &= \sum \left\{ \sum \left\{ \left[ \frac{Q_0}{\Omega \psi} \left[ G_0(t) - A_1(t) \right] \right] G(t) + \varepsilon \right\} dt \right\}
\end{align*} \tag{15}
\]
Fig. 2: Energy-dependent neutron flux profile

If we suppose that the integral
$$\int_{0}^{\infty} \int_{\Omega} \varphi(r) dr \, d\Omega$$
is convergent and taking into account the characteristics of a given nuclear reactor with spherical symmetry
$$r = \text{constant},$$
within radial range $[0, R]$:

$$\left. \begin{aligned}
&B(\vec{r})|_{\partial \Omega} = k_1 \\
&\frac{d}{dr} B(\vec{r})|_{r=0} = 0 \\
&B(\vec{r})|_{r=R} = 0 \\
&\frac{d}{dr} B(\vec{r})|_{r=R} = k_2
\end{aligned} \right\} \tag{18}$$

where, $k_1, k_2$ are core reactor characteristic constants.

For solving Eq. 16-17, the Boubaker Polynomials Expansion Scheme BPES is proposed. This scheme is applied through setting the expression:

$$B(\vec{r}) = \sum_{k=0}^{N_0} \lambda_k \times B_{4k}(\frac{r}{R}) \mu_k$$

which minimizes the Minimum Square functional $\Psi_{N_0}$:

$$\Psi_{N_0} = \sum_{k=0}^{N_0} \lambda_k \times (-2) + \sum_{k=0}^{N_0} \lambda_k \times H_k = 2k_2N_0$$

where $B_{4k}$ are the $4k$-order Boubaker polynomials, $r$ is the radius ($r \in [0, R]$), $\mu_k$ are $B_{4k}$ minimal positive roots, $N_0$ is a prefixed integer and $\lambda_k |_{k=1..N_0}$ are unknown pondering real coefficients.

At this stage, the main advantage of this step lies in Eq. 19 which ensures verifying the four boundary conditions in Eq. 18, at the earliest stage of resolution protocol. In fact, due to the properties of the Boubaker polynomials (Ghanouchi et al., 2007; Awojoyogbe and Boubaker, 2009; Labiad and Boubaker, 2007; Slama et al., 2009; Hossein et al., 2009; Fradjine and Amlouk, 2009; Belhadj et al., 2009a; 2009b, Barry and Hennessy, 2010; Agida and Kumar, 2010; Yildirim et al., 2010; Kumar, 2010; Milgram, 2011) and since $\mu_k |_{k=1..N_0}$ are roots of $B_{4k}$, Eq. 15 is reduced to Eq. 20:

$$\sum_{k=0}^{N_0} \lambda_k \times \frac{d}{dr} B_{4k}(\frac{r}{R}) \mu_k|_{r=0} = \sum_{k=0}^{N_0} \lambda_k \times 0 = 0$$

The solution is then assigned to the set of pondering real coefficients $\lambda_k |_{k=1..N_0}$ which minimizes the Minimum Square functional $\Psi_{N_0}$:

$$\Psi_{N_0} = \left( \sum_{k=0}^{N_0} \lambda_k \times (-2) \right) + \left( \sum_{k=0}^{N_0} \lambda_k \times H_k \right) = 2k_2N_0$$

With $P_3$ constant.

From Eq. 22 and our earlier assumptions, we write Eq. 23:

$$\Psi_{N_0} (\vec{r}, \Omega, E) = P_3 \psi_{N_0} (\Omega) \psi_{N_0} (E) \int \frac{1}{2N_0} \lambda_k \times B_{4k}(\frac{r}{R}) \mu_k \, d\Omega$$

$$\psi_{N_0} (\vec{r}, \Omega, E) = P_3 e^{\psi_{N_0} (\Omega) \psi_{N_0} (E)}$$

$$\psi_{N_0} (\vec{r}, \Omega, E) = P_3 e^{\psi_{N_0} (\Omega) \psi_{N_0} (E)}$$

$$\psi_{N_0} (\vec{r}, \Omega, E) = P_3 e^{\psi_{N_0} (\Omega) \psi_{N_0} (E)}$$

$$\psi_{N_0} (\vec{r}, \Omega, E) = P_3 e^{\psi_{N_0} (\Omega) \psi_{N_0} (E)}$$
Figure 2 presents the profile of the energy-dependent neutron flux as per Eq. 24. The obtained patterns are in good agreement with the results recorded by Lent et al. (2004); Zeyad et al. (2007) and Zhao et al. (2006).

CONCLUSION

An analytical solution of the neutron transport equation applying the Boubaker Polynomial Expansion Scheme BPES has been presented. The solution plots and main features have shown a good agreement with some recently published results and should present a relevant supply to studies on reactor modeling. Effects of power noise, special random effects and variable boundary conditions are subjects of future studies.

REFERENCES


