A Kind of Intersection Graphs on Ideals of a Ring

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Abstract: Problem statement: Let R be a ring. The graph G(R) is the graph whose vertices are nontrivial ideal of R and in which two vertices u, v are joined by an edge, if and only if u ∩ v ≠ {0}. Approach: In this study we study some properties of G(R). Results: We obtain conditions of R such that G(R) is a path and determine the graph G(R) in which it is a tree. Conclusion: We conclude that ideals of R have degree one.

Key words: Graphs related, intersection graph, integers modulo, algebraic structure, distinct vertices, intersection graphs, obtain conditions

INTRODUCTION

Let A = {S_i: i ∈ I} be an arbitrary family of sets. The intersection graph G (A) is the graph whose vertices are S_i, i ∈ I and in which the vertices S_i, S_j are adjacent if and only if S_i ≠ S_j and S_i ∩ S_j ≠ ∅. It is more interesting to study the intersection graphs G(A) when the elements of A have an algebraic structure. These interdisciplinary studies allow us to obtain characterization and representation of especial classes of algebraic structure in terms of graphs and vice versa.

Many authors studied such graphs related to the group structure, for example Shen (2010) and Zelinka (1975). Various construction of graphs relative to the ring structure are found in Simis et al. (1994). Chakrabarty et al. (2009) studied the intersection graphs of ideal of rings. They determined the values of n for which G(Z_n) for ring Z_n of integers modulo n for n ∈ N is connected, complete, bipartite, Eulerian and Hamiltonian.

For a given graph H, the degree of a vertex v in H denoted by deg (v), is the number of edges incident to v. A path P is a sequence of distinct vertices v_1, v_2, ..., v_m+1 in which every two consecutive vertices are adjacent. The number m is called the length of P. For two vertices u and v in a graph H the distance between u and v, denoted by d(u,v), is the length of the shortest path between u and v, if such a path exist; otherwise we define d(u,v) = ∞.

A graph H is connected if there is a path between each pair of the vertices of H. The diameter diam (H) of a connected graph H is the maximum of d(u,v) over all vertices u, v. A graph H is a tree if it be connected and have no cycle. A graph H with vertex set Eq. 1:

\{u, v_1, ..., v_m\} \hspace{1cm} (1)

And edge set E=\{\{u,v_i\}:1 ≤ i ≤ n\} is called a star graph.

In this study, we determine conditions on ring R such that G(R) is a tree or a path and get ideals of R who’s these degree is one.

MATERIALS AND METHODS

Consider the ring Z_n of integers modulo n for n ∈ N. We know that Z_n is a principle ideal ring and each of these ideals is generated by m ∈ Z_n where m is a factor of n. Let M denoted the set of all positive integers greater than one and which are not primes. In the following we have:

Theorem 1: [1] Let n ∈ M. The graph G (Z_n) is complete if and only if n = p^m where P is a prime number and m ∈ N, m>1.

We investigate the diameter G (Z_n).

Obviously we have the following theorem:

Theorem 2.1. For any n ∈ M, diam (G(Z_n))≤2

So clearly by Theorem 2.1 we have the following corollary

Corollary 2.2. Let n ∈ M, then diam (G(Z_n)) = 2 if and only if n ≠ p^m
RESULTS

We state the essential results of this study.

Proposition 3.1: Let I be a minimal ideal of a ring R. Let graph G(R) have no cycle of length 3. Then we have deg(I)=1 in G(R).

Proof: Let I be a minimal ideal of a ring R and deg(I) \geq 2. Let J and K are two distinct vertices of G(R) such that both J and K are adjacent with vertex I. Then I ∩ J ≠ (0) and I ∩ K ≠ (0). Thus I ∩ J = I, because I ∩ J ⊆ I and I is minimal. Hence I ⊆ J. Similarly we get I ⊆ k. Thus I⊆J∩k and this implies that J and K are adjacent. Thus the sequence I, J, K of vertices of G(R) is a cycle of length 3, a contradiction.

Theorem 3.2: For ring G(R) = P_2, if and only if R have only two ideal, namely minimal ideal and maximal ideal.

Proof: Let G(R) = P_2 and J,K are two vertices of G(R) Then I ∩ J = I or I ∩ J ≠ J. If I ∩ J = I, then I ⊆ J . This implies that I is minimal ideal and J is maximal ideal. If I ∩ J ≠ J, similarly we conclude that J is minimal ideal and I is maximal ideal. Conversely, for nontrivial graph G(R) is obvious.

Example: For ring R= Z_{p^3} , there is only two nontrivial ideal I = (a), J= (a^2) and we have G(R) = P_2.

Lemma 3.3: Let I be a vertex of G(R) such that deg (I)=1. Then I is a minimal ideal or maximal ideal.

Proof: Let for vertex I, deg(I)=1 and J be only vertex of G(R) such that J is adjacent to I. Then by definition, I ∩ J ≠ (0). Since deg(I)=1, I ∩ J = I or I ∩ J = J. Thus I⊆J or J⊆I. If I⊆J, then there is not nontrivial ideal L such that L ⊆ I, because, deg(I)=1. Hence I is minimal ideal. If J ⊆ I , then there is not ideal L such that L ⊆ R, because deg(I) = 1. Thus I is maximal.

Lemma 3.4: Let G(R) is a path as sequence I_1, I_2...I_n. If I be a maximal ideal of R, then G(R) = P_2.

Proof: Let I be a maximal ideal. We know that I_1 ∩ I_2 ≠ I_1 or vertices I_1 ∩ I_2 and I_1 are adjacent. If I_1 ∩ I_2 = I_1, then I_2 = I_1, because I_1 ⊆ I_2 and I_1 is maximal. This is a contradiction with I_1 ≠ I_2. Thus I_1 ∩ I_2 = I_2 and so I_2 ⊆ I_1. Let n ≥ 3. Therefore (0)≠ I_2 ∩ I_1 = I_2 or vertices I_2 ∩ I_1 and I_2 are adjacent. If (0)≠ I_2 ∩ I_1 = I_2, then I_2 ⊆ I_1 and so I_2 ⊆ I_3 ∩ I_1. Thus vertices I_1 and I_3 are adjacent, a contradiction. Otherwise we have I_2 ∩ I_3 = I_1 or I_2 ∩ I_3 = I_1. But if I_2 ∩ I_3 = I_1, then I_1 and I_3 are adjacent, a contradiction.

Theorem 3.5: Let G(R) is a path, then G(R) ≅ P_2 or G(R) = P_3. If G(R) = P_3, then R have only three ideals I_1,I_2,I_3 such that I_2=I_1 ∩ I_3.

Proof: Let G(R) be a path as sequence I_1,I_2,...I_n. By Lemma 3.3, the ideal I_1 is a minimal ideal or maximal ideal. Let I_1 be a minimal ideal. We have I_1 ∩ I_2 ≠ (0), because vertices I_1 and I_2 are adjacent. Thus I_2 ∩ I_3 = I_1, because I_1 ∩ I_2 ⊆ I_1 and I_1 is a minimal ideal and hence I_1 ⊆ I_2. Let n>2, then I_2 ∩ I_3 = I_2 implies I_3⊆I_1 and in view of I_1⊆I_2, we conclude that I_1 and I_2 are adjacent, that is a contradiction. Hence I_1 ∩ I_2 = I_1 or I_2 ∩ I_3 = I_3. If I_2 ∩ I_1 = I_2, then I_1 and I_3 are adjacent, a contradiction. Thus I_2 ∩ I_3 = I_1 and so I_2 ⊆ I_3. Thus this is true for n=3. Let n>3. If I_1 ∩ I_2 = I_1, in view of I_1⊆I_2, we conclude that I_1 and I_2 are adjacent, a contradiction. Thus vertices I_1 ∩ I_2 and I_1 are adjacent. So, I_1 ∩ I_3 = I_2 or I_1 ∩ I_3 = I_3. If I_1 ∩ I_3 = I_2 then I_3⊆I_1 and so I_3 ⊆ I_2 ⊆ I_3. This implies that I_2 and I_3 are adjacent, a contradiction. If I_1 ∩ I_3 = I_3, then I_3 ⊆ I_2 ⊆ I_3. Thus I_1 and I_3 are adjacent, a contradiction. In final, by conditions I_1⊆I_2, I_2⊆I_3 and this fact that I_1 and I_3 are not adjacent we conclude I_3=I_1 ∩ I_2.

By a similar argument as in the proof of Theorem 3.5 we see that in a tree, length of a path with end points of degree 1 is at most 2. Thus we obtain the following Corollary:

Corollary 3.6: For a ring R, the graph G(R) is a tree if and only if is a star graph.

DISCUSSION

We studied a especial class of a algebraic structure (the ring structure) in terms of graphs and vice versa. Using the properties of ideals in a ring, we discussed some of the properties of the graph G(R).

CONCLUSION

Using notions graph theory and group theory we characterized the graph G(R), when it is a path or
tree, determined ideals $I$, when deg $(I) = 1$ in the graph $G(R)$. Also we obtained a bound for $\text{diam} \ (G(Z_n))$.

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**REFERENCES**


