Complete Convergence of Exchangeable Sequences

George Stoica
Department of Mathematical Sciences,
University of New Brunswick, Saint John NB, Canada

Abstract: We prove that exchangeable sequences converge completely in the Baum-Katz sense under the same conditions as i.i.d. sequences do. Problem statement: The research was needed as the rate of convergence in the law of large numbers for exchangeable sequences was previously obtained under restricted hypotheses. Approach: We applied powerful techniques involving inequalities for independent sequences of random variables. Results: We obtained the maximal rate of convergence and provided an example to show that our findings are sharp. Conclusion/Recommendations: The technique used in the paper may be adapted in the similar study for identically distributed sequences.

Key words: Exchangeable sequences, rate of convergence, strong law of large numbers

INTRODUCTION

A sequence of random variables \{X_n\}_{n \geq 1} on the probability space \((\Omega, \mathcal{F}, P)\) is called exchangeable if for every n:

\[ P[X_1 \leq x_1, \ldots, X_n \leq x_n] = P[X_{\pi_1} \leq x_1, \ldots, X_{\pi_n} \leq x_n] \]

for any permutation \(\pi\) of \{1, 2, \ldots, n\} and any \(x_i \in \mathbb{R}\), \(i = 1, \ldots, n\). In particular, exchangeable sequences are identically distributed and one can say that future samples behave like earlier samples, or any order of a finite number of samples is equally likely. Sampling without replacement, weighted averages of i.i.d. sequences, \{Y_+\varepsilon_n\}_{n \geq 1} and \{Y_\varepsilon_n\}_{n \geq 1} are examples of exchangeable sequences, where \{\varepsilon_n\}_{n \geq 1} are i.i.d. and independent of the random variable Y. By [Chow and Teicher, 2003, Theorem 7.3.3], called de Finetti's theorem, an exchangeable sequence \{X_n\}_{n \geq 1} is conditionally i.i.d. given either the tail \(\sigma\)-field of \{X_n\}_{n \geq 1} or the \(\sigma\)-field G of permutable events.

MATERIALS AND METHODS

Under appropriate moment conditions, strong laws of large numbers for exchangeable sequences have been obtained in (Taylor and Hu, 1987; Etemadi and Kaminski, 1996; Etemadi, 2006; Rosalsky and Stoica, 2010). The rate of convergence in the above strong laws has not been obtained in full generality; for instance, papers (Zhao, 2004) assume \(p = 2\), \(r = 1\) and exponential, fourth and third order moments, respectively, for \{X_n\}_{n \geq 1}, whereas (Taylor and Hu, 1987) requires symmetry of the \(X_n\)s and obtains estimate (1) provided \(p = 2r\), \(2 \leq p < 4\). Using appropriate inequalities for independent sequences, the purpose of this study is to prove that exchangeable sequences converge completely in the Baum-Katz sense under the same conditions as i.i.d. sequences do.

RESULTS AND DISCUSSION

Theorem 1: Let \{X_n\}_{n \geq 1} be a sequence of exchangeable random variables with \(E(X_1) = 0\) and \(E|X_1|^p < \infty\) for some \(p \geq 1\). If \(0 < r < 2\), \(p \geq \max\{r, 1\}\) and \(S_n := X_1 + \ldots + X_n\), then Eq. 1:

\[ \sum_{n=1}^{\infty} n^{p/r} P(|S_n| \geq n^{1/r}) < \infty \]  

The following result (cf. (Petrov, 1995)) will be used in the proof of Theorem 1.

Lemma 2: Let \{\xi_n\}_{n \geq 1} a sequence of independent random variables with \(E(\xi_1) = 0\) and \(E|\xi_i|^p < \infty\) for all \(i \geq 1\) and some \(p \geq 1\). If \(0 < r < 2\) and \(T_n := \xi_1 + \ldots + \xi_n\), then:

\[ P(|T_n| \geq n^{1/r}) \leq C_n^{-p/r} \sum_{i=1}^{\infty} E|\xi_i|^p \]

if \(1 \leq p \leq 2\) (von Bahr – Esseen)

\[ P(|T_n| \geq n^{1/r}) \leq C_n^{-p/r} \sum_{i=1}^{\infty} E|\xi_i|^p + C \exp(-C n^{2/r} \sigma^2) \]

if \(p \geq 2\) (Fuk – Nagaev)
Where:

\[
\sigma^2 = \sum_{i=1}^{n} E(\xi_i^2)
\]

**Proof of Theorem 1:** By [(Chow and Teicher, 2003), Corollary 7.3.5] there exists a regular conditional distribution \( P^\omega \) given the \( \sigma \)-field \( \mathcal{G} \) such that for each \( \omega \in \Omega \) the mixands \( \{\xi_1 = \xi^*_n n \geq 1\} \), i.e., the coordinate random variables of the Borel probability space \((\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), P^\omega)\) are i.i.d. Namely, for all \( n \in \mathbb{N} \), any Borel function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and Borel set \( B \) on \( \mathbb{R} \), one has Eq. 2:

\[
P(\{f(X_1, ..., X_n) \in B\} = \int_{\mathbb{R}^n} P^\omega(f(\xi_1, ..., \xi_n) \in B) \, d\mathbf{P}
\]

In what follows we shall use the following notations:

\[
T_n' = \xi_n' + ... + \xi^*_n;
T_n = \xi_n(\xi_n' \geq n^{1/r}) + ... + \xi^*_n(\xi^*_n \geq n^{1/r});
\xi_n = \xi_n(\xi_n' < n^{1/r}) + ... + \xi^*_n(\xi^*_n < n^{1/r}), \text{for } n \geq 1
\]

According to (2) and the bounded convergence theorem, we have Eq. 3:

\[
\sum_{n=1}^{\infty} n^{p-1}P(|S_n| \geq n^{1/r}) = \int_0^\infty \sum_{n=1}^{\infty} n^{p-1}P(|T_n'| \geq n^{1/r}) \, d\mathbf{P}
\]

On one hand, using that \( \sum_{n=1}^{\infty} n^{p-1} \leq Ck^{p/r} \), we obtain:

\[
\sum_{n=1}^{\infty} n^{p-1}P(|\xi_n' | \geq n^{1/r}) \leq \sum_{k=1}^{\infty} kP(|k \leq \xi_n' | \leq k + 1) \sum_{n=1}^{\infty} n^{p-1} \\
\leq C \sum_{k=1}^{\infty} k^{p/r}P(|k \leq \xi_n' | \leq k + 1) \leq CE^\omega(\xi_n'^{p/r}) a.s
\]

where, \( E^\omega \) denotes expectation under \( P^\omega \). Therefore:

\[
\int_0^\infty \sum_{n=1}^{\infty} n^{p-1}P(|T_n'| \geq n^{1/r}) \, d\mathbf{P} \leq \\
\int_0^\infty \sum_{n=1}^{\infty} n^{p-1}P(|\xi_n' | \geq n^{1/r}) \, d\mathbf{P} \leq CE |X_1|^{p < \infty}
\]

On the other hand, by Lemma 2 we obtain:

\[
\sum_{n=1}^{\infty} n^{p-2}P(|T_n'| \geq n^{1/r}) \\
\leq C \sum_{n=1}^{\infty} n^{-2} \sum_{k=1}^{n} E^\omega(|\xi_n' | \leq n^{1/r}) + C \sum_{n=1}^{\infty} n^{p-2} \\
\exp\left\{-Cn^{p-2}(|\xi_n' | \leq n^{1/r})\right\} \\
\leq C \sum_{n=1}^{\infty} E^\omega(|\xi_n' | \leq \sum_{j=1}^{n} j^2) + C \sum_{n=1}^{\infty} n^{p-2} \exp\left\{-Cn^{2p-2}E^\omega(\xi_n'^{p/r})\right\} \\
\leq CE^\omega(\xi_n'^{p/r}) + C \sum_{n=1}^{\infty} n^{p-2} \exp\{-Cn^{2p-1}\} a.s
\]

The latter series is convergent as \( r < 2 \). Therefore:

\[
\int_0^\infty \sum_{n=1}^{\infty} n^{p-2}P(|T_n'| \geq n^{1/r}) \, d\mathbf{P} \leq CE |X_1|^{p + 1} < \infty
\]

Conclusion (1) now follows from (4) and (5) via (3).

**CONCLUSION**

It is very well known that we cannot allow \( p < 1 \) in the Baum-Katz estimate (1) for i.i.d. sequences; the following example shows that the same is true for exchangeable sequences. Consider \( X_n = Y.\varepsilon_n \), where \( \{\varepsilon_n\}_{n \geq 1} \) are i.i.d. and independent of a Cauchy random variable \( Y \), with \( P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2 \). We have \( E|X_1|^p < \infty \) for all \( 0 < p < 1 \), but \( E|X_1| = \infty \). As \( X_1 \sim -X_1 \), we have:

\[
\frac{S_n}{n} = Y - \sum_{i=1}^{n} \varepsilon_i \rightarrow 0 \text{ a.s}
\]

i.e., the strong law of large numbers holds for the exchangeable sequence \( \{X_n\}_{n \geq 1} \). On the other hand:

\[
\sum_{n=1}^{\infty} n^{p-2}P(|S_n| \geq n^{1/r}) = \sum_{n=1}^{\infty} n^{p-2} \\
P(|\varepsilon_1 + ... + \varepsilon_n| \geq n).P(|Y| \geq n^{1/r}) \\
\leq C \sum_{n=1}^{\infty} n^{p-2} \int_{0}^{\infty} \frac{1}{x^{p/r}} \, dx
\]

which diverges for all \( p \geq r \) and \( 0 < r < 2 \).

**REFERENCES**


