Edge Double-Critical Graphs

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Abstract: Problem statement: The vertex double-critical conjecture that the only vertex double-critical graph is the complete graph has remained unresolved for over forty years. The edge analogue of this conjecture has been proved. Approach: It was observed that if the chromatic number decreases by two upon the removal of a 2-matching, then the 2-matching comprises four vertices which determine an induced subgraph isomorphic to the complete graph on four vertices. This observation was generalized to t-matchings. Results: In this note, it has been shown that the only edge double-critical graph is the complete graph. Conclusion/Recommendations: An alternate proof that the only edge double-critical graph is the complete graph has been obtained. Moreover, the result has been obtained independently.

Key words: Chromatic number, critical clique, k-matching

INTRODUCTION

The graphs considered in this study are finite, undirected and simple. For a given graph G, the vertex and edge sets of G are denoted by V(G) and E(G), respectively. The order of G, denoted by n = |V(G)|, is the cardinality of V(G). An r-clique is a complete subgraph of order r and is denoted by K_r. A subset M of E(G) is said to be independent whenever no two edges in M share a common vertex. In case |M| = k, the set M is called a k-matching. For a subset X of V(G), the subgraph of G induced by X is denoted by G[X]. All vertex colorings are proper, i.e., a partition of V(G) into independent subsets of V(G) called color classes. Lastly, \( \chi(G) \) denotes the chromatic number of G and is the minimum cardinality of a partition of V(G) determined by a proper vertex coloring of G.

A graph G is said to be vertex double-critical provided \( \chi(G-v) = \chi(G)-2 \) for every adjacent pair of vertices u, v. This definition arises out of its relation to the Erdos-Lovasz Tihany Conjecture. A special case of this conjecture is that the only vertex double-critical graph is the complete graph; it is often referred to as the Erdos-Lovasz double-critical conjecture. Relations to FTTMs and the inertia tensor of a tetrahedron as defined in (Ahmad et al., 2010; Tonon, 2005), respectively are also being investigated.

Edge double-critical graphs: It is now shown that \( K_n \) is the only edge double-critical graph. First, some notational conventions and a required definition are given. Let \( M_2 = \{e_1, e_2, \ldots, e_t\} \) be a set of t edges in E(G) and set \( e_i = u_iv_i \) for i = 1, 2, \ldots, t. Next, define \( M'_2 = \{u_1, v_1\} \cup \{u_2, v_2\} \cup \cdots \cup \{u_t, v_t\} \). Clearly, \( M_t \) is a t-matching when \(|M_t| = 2t\).

Definition 1: Let G be a graph which contains 2-matchings. Then G is called edge double-critical whenever \( \chi(G-M_2) = \chi(G)-2 \) for every 2-matching \( M_2 \).

Necessarily, an edge double-critical graph is connected. An important observation is given in the following lemma.

Lemma 1: Let \( M_2 = \{e_1, e_2\} \) be a 2-matching such that \( \chi(G-M_2) = \chi(G)-2 \). Then \( G[M'_2] \cong K_4 \).

Proof: Set \( k = \chi(G) \) and let \( e_i = u_iv_i \) for i = 1, 2. Consider any (k-2)-coloring of G-\( M_2 \), the colors being from among \( \{c_1, c_2, \ldots, c_k-2\} \). Then \( u_1 \) and \( v_1 \) must be colored the same since otherwise there would exist a (k-2)-coloring of G-\( e_2 \). A similar argument shows that \( u_2 \) and \( v_2 \) must be colored the same, necessarily using a different color from that used for \( u_1 \) and \( v_1 \). Next,
observe that \( u_1, u_2 \in E(G-M_2) \). Else, both \( u_1 \) and \( u_2 \) could be recolored using color \( c_{k-1} \). But this would allow \( e_1 \) and \( e_2 \) to be added back to \( G-M_2 \) resulting in a coloring of \( G \) using fewer than \( k \) colors. A similar argument shows that \( u \) and \( v \) can be added back to \( G-M_2 \) resulting in a coloring of \( G \) using fewer than \( k \) colors. Furthermore, \( G[M_2'] \cong K_4 \).

**Theorem 1:** Let \( t \geq 1 \). If \( \chi(G-M_1) = \chi(G) \) then \( M_1 \) is a \( t \)-matching of \( G \). Moreover, \( G[M_1] \cong K_2t \).

**Proof:** Let \( k = \chi(G) \). The result is trivial for \( t = 1 \). Let \( t \geq 2 \) and consider a subset \( M_t \) of \( E(G) \) such that \( \chi(G-M_t) = k-t \). Because \( M_t = t \), it follows that \( M_t \) is a \( t \)-matching as incident edges can decrease the chromatic number of a graph by at most one upon their removal. Observe now that for all pairs \( i, j \) with \( i \neq j \), \( \chi(G-e_i-e_j) = k-2 \). By setting \( M_2 = \{ e_i, e_j \} \) and applying Lemma 1, \( G[M_2] \cong K_4 \). Hence, \( G[M_1] \cong K_2t \).

**Proposition 1:** Every \( t \)-matching in \( K_{2t} \) is critical.

**Proof:** The proof is by induction on \( t \). For \( t = 1 \), the result is trivial. Since \( \chi(K_{2t} - M) = \chi(C_4) = 2 \) for every 2-matching \( M \) of \( K_4 \), Proposition 1 holds for \( t = 2 \). Now, inductively assume that Proposition 1 holds for \( t = 1, 2, \ldots, t'-1 \). Let \( M_t \) be any \( t' \)-matching in \( K_{2t} \). Notice that \( K_{2t'} \) can be written as \( K_{2t'} = K_t + K_{2(t'-1)} \). Moreover, it can be assumed, without loss of generality, that the single edge in the \( K_2 \) term is in the \( t' \)-matching \( M_t \). Consequently, \( M_t \) can be written as \( M_t = M_{t-1} \cup M_{t-1}' \), where \( M_t = K_{2t-1} \) and \( M_{t-1}' \) is a \( (t'-1) \)-matching in the \( K_{2(t'-1)} \) term. By the inductive hypothesis, \( \chi(K_{2(t'-1)} - M_{t-1}') = t'-1 \). Therefore:

\[
\chi(K_{2t'} - M_t) = (K_{2t} - K_{2(t'-1)}) - (M_{t-1} \cup M_{t-1}') = (K_2 - M_2) + (K_{2(t'-1)} - M_{t-1}') = 1 + (t'-1) = t'.
\]

Hence, \( \chi(K_{2t'} - M_t) = 1 + (t'-1) = t' \).

**Corollary 1:** Every matching in \( K_n \), \( n \geq 2 \), is critical.

Lemma 1 and Corollary 1 together set the stage for the main result of this note.

**Theorem 2:** \( G \) is edge double-critical if and only if \( G \cong K_n \) provided \( n \geq 4 \).

**Proof:** If \( G \cong K_n \), where \( n \geq 4 \), then by Corollary 1, every 2-matching in \( K_n \) is critical. Thus, \( G \) is edge double-critical. Conversely, let \( G \) by a connected, edge double-critical graph. Take any \( u, v \in V(G) \) and suppose to the contrary that \( uv \notin E(G) \). Then \( N(u) = N(v) = \{ w_{u,v} \} \) for some vertex \( w_{u,v} \in V(G) \). Otherwise, because \( G \) is connected, it would follow that \( u, v \in M_2' \) for some 2-matching \( M_2 \). Since \( G \) is edge double-critical, \( G[M_2'] \cong K_4 \) by Lemma 1. This implies that \( uv \in E(G) \), contrary to our supposition. Next, observe that \( N(z) = \{ w_{u,v} \} \) for every vertex \( z \neq w_{u,v} \). Else, by using exactly the same argument as above, we would be forced to conclude that \( z \in N(u) = \{ w_{u,v} \} \), which is clearly not possible by the choice of \( z \). The above argument leads to the conclusion that \( G \) is a star. But such a graph is known not to be edge double-critical because of the absence of 2-matchings in any star. Hence, \( uv \notin E(G) \) so that \( G \cong K_n \).

REFERENCES


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