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Abstract Fixed Points of Set-valued Mappings

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Abstract: We defined new concept of complete sequences of equivalence relations on a given abstract set on which no topology is considered, the concept of contraction mappings with respect to these sequences, proved the existence of a fixed point of a multivalued and in particular single valued mappings T on the given abstract set.

Key words: Abstract fixed points, set-valued mappings, topology

INTRODUCTION

Let X be a metric space with metric d. Denote by CB(X) the class of all nonempty closed bounded subsets of X and define the Hausdorff metric H by

$$H(A, B) := \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\} \text{ for } A, B \in CB(X).$$

A mapping T of a metric space (X, d) into itself is α- contraction for some $\alpha \in (0, 1)$ if and only if $d(T(x), T(y)) \le \alpha\ d(x, y)$ for all $x, y \in X$.

A mapping T of a metric space (X, d) into CB(X) is α-contractive for some $\alpha \in (0, 1)$ if and only if $H(T(x), T(y)) \le \alpha\, d(x, y)$ for all $x, y \in X$.

The Banach contraction principle stated that if (X, d) is a complete metric space and let T be a contraction mapping from X into itself. Then T has a unique element $x \in X$ such that $T(x) = x$. Moreover the sequence $\{T^n(y)\}_{n\in N}$ is converging strongly to x for every $y\in X$.

Nadler[1] generalized this principle as follows: Let (X, d) be a complete metric space and T be α-contractive mapping from X into CB(X). Then there exists $y\in X$ with $y\in T(y)$. For other results[2-4].

In this paper we introduce the concept of complete sequences of equivalence relations on a given abstract set on which no topology is considered, the concept of contraction mappings with respect to these sequences, proved the existence of a fixed point of a multivalued and in particular single valued mappings T on such a given abstract set. we also showed that the existence of complete sequence of equivalence relations on a given abstract set is restricted to the existence of a non-Archimedean metric on it and the multivalued mappings are contraction with respect to this sequence of equivalence relation and its contractive condition with respect to the corresponding metric are equivalent in that case the sequence of iterates $T^n(y)$ is Mosco's convergent to the set of fixed points of T for every y in X. We supported our work with some examples, showed the validation of Nadler's fixed point theorem for multivalued mappings.

A sequence of subsets $\{A_n\}_{n=0}^{\infty}$ in a metric space (X, D) is said to be Mosco's convergent to a subset A if and only if for every $x \in A$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in A_n\ \forall\ n \in N$ such that $\{x_n\}_{n=1}^{\infty}$ is strongly convergent to the point x[5].

A metric d on a set X is call non-Archimedean if and only if for all $x, y, z \in X$, $d(x, y) \le \max\{d(x, z), d(z, y)\}$, then, in fact $d(x, y) = \max\{d(x, z), d(z, y)\}$.

If $d(x, z) \neq d(z, y)$ and therefore, each non-Archimedean metric space has the geometric property that each three points of it are vertices of an isosceles triangle[6].

We introduce the following definitions.

Definitions

1. Let X be an abstract set. Then the sequence of equivalence relations $\{R_n\}_{n=0}^{\infty}$ on X is said to be complete if and only if $\{R_n\}_{n=0}^{\infty}$ fulfills the following conditions.

i. $X\times X = R_0 \supseteq R_1 \supseteq R_2 \ldots$;

ii. If $A, B \subseteq X$, $A\times B \subseteq \bigcap_{n=0}^{\infty} R_n$, then either $A \subseteq B$ or $B \subseteq A$,

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iii. If a sequence $\{A_n\}_{n=0}^{\infty}$ of P(X), the set of all non-empty subset of X, satisfies $A_n \times A_{n+1} \subseteq R_n$ for all $n \in N_0$, then there exists a non empty set A such that $A_n \times A \subset R_n\ \forall n \in N$.

2. Let X be an abstract set. Then the sequence of equivalence relations $\{R_n\}_{n=0}^{\infty}$ on X is said to be diagonally complete if and only if $\{R_n\}_{n=0}^{\infty}$ fulfills the following conditions

i. $X \times X = R_0 \supseteq R_1 \supseteq R_2 \ldots$;

ii. $\bigcap_{n=0}^{\infty} R_n = \Delta$, the diagonal in $X \times X$;

iii. For a sequence $\{A_n\}_{n=0}^{\infty}$ of P(X) such that $A_n \times A_{n+1} \subseteq R_n$ for all $n \in N_0$, there exists $x \in X$ such that $A_n \times \{x\} \subseteq R_n$ for all $n \in N_0$.

3. Let X be an abstract set, $\{R_n\}_{n=0}^{\infty}$ be a complete sequence of equivalence relations on X. A mapping T:X $\rightarrow$P(X), set-valued mapping or a single valued, is said to be contraction with respect to $\{R_n\}_{n=0}^{\infty}$ iff T satisfies the assumption, given any two subsets $A, B \in P(X)$ with $A \times B \subseteq R_n$, $n \in N_0$, then $T(A) \times T(B) \subseteq R_{n+1}$.

Remarks

1. If $\{R_n\}_{n=0}^{\infty}$ is a complete sequence of equivalence relations on X, we in particular have,

a. $\{x\}x\{y\} \subseteq \bigcap_{n=0}^{\infty} R_n$ if and only if x=y.

b. If $\{x_n\}_{n=0}^{\infty}$ is a sequence in X, such that $(x_n, x_{n+1}) \in R_n$ for all $n \in N_0$, then there is x in X such that $(x_n, x) \in R_n\ \forall\ n \in N$.

2. If T is multivalued and δ (T(x), T(y)) := Sup {d(a, b): a ∈T(x), b∈T(y)}, we have H(T(x), T(y)) ≤ δ (T(x), T(y)) for every x, y ∈X and x ≠ y, hence if T satisfies the condition δ (T(x), T(y) ≤ α d(x, y) for every x, y ∈X and x ≠ y, then T is α-contractive and according to Nadler's theorem T has fixed point.

We have the following Results.

Theorem 1: Let X be an abstract set and $\{R_n\}_{n=0}^{\infty}$ be a complete sequence of equivalence relations on X. If T:X $\rightarrow$P(X) is a contraction set-valued mapping with respect to $\{R_n\}_{n=0}^{\infty}$ from X into P(X), then T has fixed points $x^* \in X$. Moreover, if F(T) is the set of fixed points of T, $T^n(y) \times F(T) \subseteq R_n$ for each $y \in X$ and $n \in N_0$.

Proof: Fix $y \in X$ and use (i), we see that $\{y\} \times T(y) \subseteq R_0$, since T is contraction with respect to $\{R_n\}_{n=0}^{\infty}$, we may infer that

$$T^n(y) \times T^{n+1}(y) \subseteq R_n\ \forall\ n \in N_0. \quad (1)$$

Let $A_n = T^n(y)$, (iii) insures the existence of non empty subset A(y) of X such that $T^n(y) \times A(y) \subseteq R_n\ \forall\ n \in N_0$. We claim that every element in the set A(y) is a fixed point of T. Let $x^* \in A(y)$, since $T^n(y)$ x{x*} ⊆ $T^n(y)$x A(y), we see that

$$T^n(y) \times \{x^*\} \subseteq R_n\ \forall\ n \in N_0. \quad (2)$$

Since T is contraction with respect to $\{R_n\}_{n=0}^{\infty}$,

$$T^{n+1}(y) \times T(x^*) \subseteq R_{n+1} \subseteq R_n\ \forall\ n \in N_0, \quad (3)$$

since R_n is transitive for all $n \in N_0$, (1) and (3) insure that,

$$T^n(y) \times T(x^*) \subseteq R_n\ \forall\ n \in N_0, \quad (4)$$

Since R_n is symmetric for all $n \in N_0$, (4) insures that,

$$T(x^*) \times T^n(y) \subseteq R_n\ \forall\ n \in N_0 \quad (5)$$

Since R_n is transitive, (2) and (5) proved that

$$T(x^*) \times \{x^*\} \subseteq R_n\ \forall\ n \in N_0,$$

equivalently, $T(x^*) \times \{x^*\} \subseteq \bigcap_{n=0}^{\infty} R_n$, using (ii), we see that $\{x^*\} \subseteq T(x^*)$ or $T(x^*) \subseteq \{x^*\}$. Either of the two cases proved that x^* is a fixed point of T, $x^* \in X$, this proved that A(y) ⊆ F(T) clearly $T^n(y) \times \{x^*\} \subseteq R_n$ for each $y \in X$, $n \in N_0$ and $x^* \in$ A(y), therefore $T^n(y) \times A(y) \subseteq R_n$ for each $y \in X$, $n \in N_0$. Finally, let z be any fixed point of T, since (y, z) $\in R_0$ and T is contraction with respect to $\{R_n\}_{n=0}^{\infty}$,

$$T^n(y) \times T^n(z) \subseteq R_n \quad \forall n \in N, \quad (6)$$

on the other side

$$T^n(z) \times A(z) \subseteq R_n \quad \forall n \in N, \quad (7)$$

Since R_n is symmetric and transitive, (6) and (7) showed that

$$T^n(y) \times A(z) \subseteq R_n \quad \forall n \in N,$$

Consequently $T^n(y) \times F(T) \subseteq R_n \quad \forall n \in N.$

Corollary 1: Let X be an abstract set and $\{R_n\}_{n=0}^{\infty}$ be a diagonally complete sequence of equivalence relations on X. If T:X $\rightarrow$P(X) is a contraction set-valued or a single valued mapping with respect to $\{R_n\}_{n=0}^{\infty}$, then T has a unique fixed point $x^* \in X$. Moreover, $T^n(y)\times\{x^*\} \subset R_n$ for every $y \in X$ and $n \in N_0$.

Proof: Fix $y \in X$ and use (i), we see that $\{y\}\times T(y) \subseteq R_0$, since T is contraction with respect to $\{R_n\}_{n=0}^{\infty}$, we may infer that

$T^n(y)\times T^{n+1}(y) \subseteq R_n \ \forall n \in N_0$.

Following the same steps of theorem1, the set A given their is a singleton x^* in X, $T(x^*)\times\{x^*\} \subseteq R_n \ \forall \ n \in N_0$, equivalently, $T(x^*)\times\{x^*\} \subseteq \bigcap_{n=0}^{\infty} R_n$. Clearly $T^n(y)\times\{x^*\} \subseteq R_n$ for each $y \in X$ and $n \in N_0$ since $\{R_n\}_{n=0}^{\infty}$ is diagonally complete, we see that $T(x^*) = \{x^*\}$. To show the uniqueness of x*, let $y^* \in X$ be such that $T(y^*) = \{y^*\}$, then $\{x^*\}\times\{y^*\} \subseteq R_0$, since T is a contraction mapping with respect to $\{R_n\}_{n=0}^{\infty}$, we see that $T^n(x^*)\times T^n(y^*) \subseteq R_n \ \forall \ n \in N_0$. Then $\{x^*\}\times\{y^*\} \subseteq R_n \ \forall \ n \in N_0$, since $\{R_n\}_{n=0}^{\infty}$ is diagonally complete, we see that $x^* = y^*$.

Examples

1. Let X be the finite set, X:={(0, 4), (0, -4), (3, 0) and (8, 0)} the following defined set of equivalence relations $\{R_0, R_1, R_2 \text{ and } R_3\}$ is diagonally complete:

R_0:=XxX

R_1:= $\Delta \cup$ {((0, 4), (3,0)), ((0,4),(0, -4)), ((3,0), (0,-4)), ((3,0), (0, 4)), ((0,-4),(3,0)), ((0, -4), (0, 4)) }

R_2:= $\Delta \cup$ {((0, 4), (3,0)), ((3,0), (0, 4)) }

R_3 := Δ

Any of the following multivalued mappings is contraction with respect to $\{R_0, R_1, R_2 \text{ and } R_3\}$

T_1:= { ((0,4),{(3,0)}), ((0,-4), {(3, 0), (0, 4)}), ((3, 0),{ (3,0)}), ((8,0), {(3,0)})}

T_2:= {((0,4),{ (3, 0), (0, 4)}), ((0,-4), {(3,0)}), ((3, 0),{ (3, 0), (0, 4)}), ((8,0),{(3, 0), (0, 4), (0, -4)})}

T_3 := { ((0,4),{ (3, 0), (0, 4)}), ((0,-4), {(0, 4)}), ((3, 0),{ (0, 4)}), ((8,0),{(3, 0)})}

T_4 := { ((0,4),{(3, 0)}), ((0,-4), {(0, 4)}), ((3, 0),{ (3, 0), (0, 4)}), ((8,0),{ (3, 0), (0, -4)})}…and many other, such mappings all have a unique fixed point, note that as X is a finite subset of the Euclidean space it is complete and bounded whose Diam X=4$\sqrt{5}$ and every three points lie on isosceles triangle.

2. Consider X = Z, the set of all integer numbers. The following sequence of equivalence relations $\{R_n\}_{n=0}^{\infty}$ on Z is diagonally complete, $R_n = \{(x,y):\ x,y \in Z,\ 2^n \text{ divides } (x-y)\}$ for all $n \in N$, we have $R_0 = Z\times Z$ and as $R_n = \{(x,y):\ x,y \in Z,\ x-y = 2^n r,\ r \in Z\}$, $\{R_n\}_{n=0}^{\infty}$ satisfies $Z\times Z = R_0 \supseteq R_1 \supseteq R_2 ... \supseteq R_m \supset ...$ .To prove that $\bigcap_{n=0}^{\infty} R_n = \Delta$, the diagonal in $Z\times Z$, let $(x,y) \in R_n \ \forall n \in N$ and $x \neq y$, then there is a sequence of integer $\{r_n\}_{n=0}^{\infty} \subset Z$ such that $x - y = 2^n r_n$, hence $\frac{1}{2^n} = \frac{r_n}{x-y}$, this in turns showed that $\{r_n\}_{n=0}^{\infty} \subset Z$ is convergent to zero sequence which is a contradiction as a sequence of integers, otherwise $\{r_n\}_{n=0}^{\infty} \subset Z$ must be stationary after certain natural number, thus there is a natural number n such that $\{r_1, r_2, \ldots r_n, r_n, r_n, \ldots\}$ and $x - y = 2^n r_n$ with $(x,y) \notin R_m \ \forall m > n$ which is a contradiction, hence x=y. Finally condition (iii) is true. Let z be an integer and consider the mapping T: Z $\rightarrow$ Z defined by T(k) = 2k+z. k$\in$Z, T is contraction with respect to $\{R_n\}_{n=0}^{\infty}$, indeed if A, B $\in$P(Z) such that $A\times B \subseteq R_n$ for all $n \in N_0$, then for each $a \in A, b \in B$ there is $r \in Z$ such that $a - b = 2^n r$. Hence T(a)-T(b) = 2a -2b =2(a-b) =2(2^n r) =2^{n+1} (r). Consequently $T(A)\times T(B) \subseteq R_{n+1}$. The unique fixed point of T is (-z) a number in Z, T(-z) =- z, since $T^n(k) - (-z) = 2^n(k+z)$, we see that $T^n(k)\times\{-z\} \subset R_n$ for each k$\in$Z and all $n \in N_0$.

We have the following interesting theorem.

Theorem 2: Let X be an abstract set. Then the following are equivalent,

1. There exists a complete sequence of equivalence relations $\{R_n\}_{n=0}^{\infty}$ on X,
2. There exists a non-Archimedean bounded and complete metric d on X.

Proof: Let $\{R_n\}_{n=0}^{\infty}$ be a complete sequence of equivalence relations on X and x, y $\in$ X are two distinct elements of X, it follows from (ii), that the set $\{n \in N_0 : (x,y) \in R_n\}$ is of a form $\{0,1,2,3,....,p\}$ for some $p \in N_0$ and if x=y, (x, y) $\in$ R_n for every n. Set

$$P(x,\ y)=\begin{cases} p, & \text{if } x \neq y,\\ \infty, & \text{if } x = y.\end{cases}$$

For each $\alpha \in (0, 1)$ define $d(x,\ y)=\alpha^{p(x,\ y)}\quad \forall\ x,\ y \in X$, we see that

i. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

ii. since the function p(.,.) is symmetric, d is symmetric.

iii. Now fix elements x, y and z in X, without loss of generality, we assume that x, y and z are distinct. Then $(x,z)\in R_{p(x,z)}$ and $(z,y)\in R_{p(z,y)}$. Set

$k=\min\{p(x,z),\ p(z,y)\}$

since $\{R_n\}_{n=0}^{\infty}$ is descending, we have $(x,z),(z,y)\in R_k$. Since

R_k is transitive, $(x,y)\in R_k$ and hence $p(x,y)\geq k$, then by the definition of k

$$\begin{aligned} d(x,y) &= \alpha^{p(x,y)} \leq \alpha^k = \max\{\alpha^{p(x,z)},\alpha^{p(z,y)}\}\\ &= \max\{d(x,z),d(z,y)\}\end{aligned}$$

which means that d is a non-Archimedean metric, clearly d is bounded as $0 < d(x,y) \leq 1\quad \forall\ x,y\in X$.

Now, we show that d is complete. Let $\{x_n\}_{n=0}^{\infty}$ be a Cauchy sequence in (X, d), then there is a subsequence $\{x_{k_n}\}_{n=0}^{\infty}$ such that $d(x_{k_n},x_{k_{n+1}})<\alpha^n$.

Set $y_n = x_{k_n}$, since $d(y_n,y_{n+1})<\alpha^n$, we get $p(y_n,y_{n+1}) > n$.

Hence by the definition of $p(y_n,y_{n+1})\in R_n$, put $A_n=\{y_n\}$, we get $A_n \times A_{n+1} \subseteq R_n$, (iii) insures the existence of an element y in X such that

$A_n \times \{y\} \subseteq R_n$, equivalently,

$(y_n, y) \in R_n\ \ n \in N_0$,

then $p(y_n, y) \geq n$ which yields that

$d(y_n, y) \leq \alpha^n$.

Hence $\{x_{k_n}\}_{n=0}^{\infty}$ is convergent as a Cauchy sequence contains a convergent subsequence, this proved that X is complete.

Conversely, let d be a non-Archimedean bounded and complete metric on X. Define $R_n = \{ (x, y) \in X \times X: d(x, y) \leq \alpha^n \operatorname{Diam}(X) \}$ for all $n\in N_0$, Diam(X) denotes the diameter of X. Then it is obvious that R_n are reflexive, symmetric, to show that R_n is transitive, let (x, y), (y, z) $\in R_n$, we have

$$\begin{aligned} d(x, y) &\leq \max\{d(x, y), d(y, z)\}\\ &= \max\{\alpha^n \operatorname{Diam}(X), \alpha^n \operatorname{Diam}(X)\}\\ &=\alpha^n \operatorname{Diam}(X),\ n \in N_0,\end{aligned}$$

which implies (x, z) $\in R_n$. Now conditions (i) is clearly true, to show that (ii) holds, let A, B $\subseteq$ X, $A\times B\subseteq\bigcap_{n=0}^{\infty}R_n$, we will show that either $A \subseteq B$ or $B \subseteq A$, other wise there are two points $a\in A$, $a\notin B$ and $b\in B$, $b\notin A$, since $(a, b) \in R_n$ for all $n\in N_0$, we see that $d(a, b) \leq \alpha^n \operatorname{Diam}(X)$, taking the limit as $n\longrightarrow\infty$ showed that d(a, b) =0, hence a=b which is a contradiction. Finally we are going to verify condition (iii), assume that a sequence $\{A_n\}_{n=0}^{\infty}$ such that $A_n \times A_{n+1} \subseteq R_n$ for all $n\in N_0$, that is $(x, y) \in R_n\ x \in A_n$, $y \in A_{n+1}$. Pick a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n\in A_n$, thus $(x_n,x_{n+1})\in R_n$ for every $n\in N_0$, therefore $d(x_n,x_{n+1})\leq\alpha^n \operatorname{Diam}(X)\ \forall n\in X$, hence we see that

$$\begin{aligned} d(x_n,x_{n+k}) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots d(x_{n+k-1},x_{n+k})\\ &\leq (\alpha^n+\alpha^{n+1}+\ldots\alpha^{n+k-1})\operatorname{Diam}(X)\\ &\leq \alpha^n\left[\frac{1}{1-\alpha}\right]\operatorname{Diam}(X)\quad \text{for every } k\in N,\end{aligned}\tag{8}$$

taking the limit as $k\longrightarrow\infty$ showed that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in a complete space (X, d), hence there exists an element $x\in X$ such that $x_n \to x$. We will prove that $(x_n, x) \in R_n$. Fix an $n\in N_0$, using (8), we see that $(x_n, x_{n+k}) \in R_n$. Now let k tends to infinity, we obtain that $(x_n, x) \in R_n$. take A to be such all limits, let $y \in A_n$ and $x \in A$ using the definition of A there is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n\in A_n$ and $(x_n,x)\in R_n$ once $(x_{n+1},x_n)\in R_n$, $(y, x_{n+1}) \in A_n \times A_{n+1} \subseteq R_n$ for all $n\in N_0$ and R_n is symmetric and transitive, we see that $(y, x) \in R_n$ for all $n\in N_0$. Equivalently $A_n \times A \subseteq R_n$. $n\in N_0$. This completes the proof of condition (iii).

Now, our main purpose is to show that Theorem 1 is equivalent to the restriction of Nadler's theorem to the class of non-Archimedean bounded metric space. We have the following interesting theorem.

Theorem 3: Let T be a set-valued mapping of an abstract set X into P(X) and $\alpha \in (0, 1)$. If $\{R_n\}_{n=0}^{\infty}$ is a complete sequence of equivalence relations in X, d(., .) is the corresponding non-Archimedean bounded and complete metric space. Then T is contraction with respect to $\{R_n\}_{n=0}^{\infty}$ if and only if T is contraction mapping in the metric space (X, d).

Proof: Let T be contraction with respect to $\{R_n\}_{n=0}^{\infty}$ and fix two distinct elements x, y in X, then $(x, y) \in R_{p(x, y)}$ using (1), $T(x) \times T(y) \subseteq R_{p(x, y)+1}$. That is $(a, b) \in R_{p(x, y)+1}$ for every $a \in T(x)$, $b \in T(y)$.

Hence $p(a, b) \geq p(x, y) + 1$ for every $a \in T(x)$, $b \in T(y)$, this implies that $d(a, b) = \alpha^{p(a, b)} \leq \alpha^{p(x, y) + 1} = \alpha\, d(x, y)$. Hence

$\delta(T(x), T(y)) = \text{Sup} \{d(a, b) : a \in T(x), b \in T(y)\}$
$\leq \alpha \, \text{Sup} \{d(a, b) : a \in T(x), b \in T(y)\}$
$= \alpha \, d(x, y)$.
Conversely, if $A, B \in P(X)$, $A \times B \subseteq R_n$, then
$(x, y) \in R_n$ for every $x \in A, y \in B$,
Since T is contraction mapping in the metric space (X, d).
$\delta(T(x), T(y)) \leq \alpha \, d(x, y) \leq \alpha^{n+1} \text{Diam}(X)$,
so for each $a \in T(x), b \in T(y)$,
$d(a, b) \leq \delta(T(x), T(y)) \leq \alpha^{n+1} \text{Diam}(X)$
implies
$(a, b) \in R_{n+1}$ for every $a \in T(x), b \in T(y)$.
So
$T(x) \times T(y) \subseteq R_{n+1}$ $x \in A, y \in B$,
Equivalently
$T(A) \times T(B) \subseteq R_{n+1}$,
This proved that T is contraction with respect to $\{R_n\}_{n=0}^{\infty}$.

The following corollary, restricted to the non-Archimedean bounded metric spaces case, fulfills our main purpose.

Corollary 2: Let $\{R_n\}_{n=0}^{\infty}$ be a complete sequence of equivalence relations in X and d(., .) is the corresponding non-Archimedean bounded and complete metric. Then the following are equivalent,

1. T has fixed points, in this case the sequence of iterates $T^n(y)$ is Mosco's convergent to the set of fixed points F(T) of T.
2. T is contraction mapping in the metric space (X, d).

Proof: We need here only to verify the suitable property of a sequence of successive approximations. Theorem1 implies Nadler's theorem follows from the fact that, if
$R_n = \{(x, y): d(x, y) \leq \alpha \, \text{Diam}(X) \}$.
Then by Theorem1, T has fixed point x^* and
$T^n (x) \times \{ x^* \} \subseteq R_n$.
That is
$d(y_n, x^*) < \alpha^n \text{Diam}(X)$ for every $y_n \in T^n(x)$
in particular, $y_n \to x^*$ for all $y_n \in T^n(x)$, so $T^n(x) \to \{ x^* \}$.
We show the converse implication, under the assumptions of Theorem1. If d is defined by
$d(x, y) = \alpha^{p(x, y)}$ $x, y \in X$.
Then by Nadler's theorem T has fixed point x^*, moreover, given $n \in N_0$ and $y \in X$,
$\delta(T^n(y), \{x^*\}) = \delta(T^n(y), T^n(x^*)) \leq \alpha^n d(y, x^*) \leq \alpha^n$.
That is
$d(y_n, x^*) \leq \alpha^n$ for every $y_n \in T^n(y)$.
Hence $p(y_n, x^*) \geq n$, which implies that
$(y_n, x) \in R_n$ for every $y_n \in T^n(y)$
Equivalently
$T^n(y) \times \{ x^*\} \subseteq R_n$.

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